The hardness of decision-tree complexity

FLAT Seminar

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The complexity of complexity

Suppose you are given a description of a computational problem P

and you wish to find the "best" algorithm for solving P

in a certain computational model M .

What is the computational complexity of this "meta" task?

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- However, if the language P is described to you by way of a *non-deterministic* finite automaton (NFA), finding the smallest NFA for computing P is PSPACE-hard (Jiang&Ravikumar, 1993).

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- However, if the language P is described to you by way of a *non-deterministic* finite automaton (NFA), finding the smallest NFA for computing P is PSPACE-hard (Jiang&Ravikumar, 1993).
- So this problem can be sometimes easy, sometimes hard, and there are many cases where the complexity of the above problem is unknown.

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- The above problem does not necessarily become harder as the model M becomes more powerful. Soon I will give examples of computational models M_1 and M_2, where M_2 is more powerful than M_1, but finding optimal algorithms is possible for M_2 but (e.g.) NP-hard for M_1.

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- The answer also depends on the kind of problems we wish to understand. Soon I will give examples where we can answer the above question efficiently, e.g., for total functions, but the meta-problem becomes NP-hard for partial functions.

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  - The kind of problem P for which we wish to find good algorithms.
  - How the problem P is described to us.

Some known results

Model	**Measure**	**Problem**	**Description**	**Hardness**	*reference*

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PP-CC	depth	total f	matrix	in P	LS'09

Our results

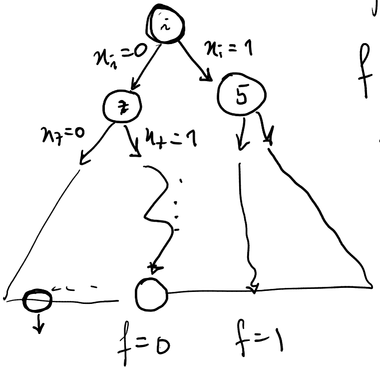
We understand the complexity of decision-tree complexity.

Decision trees

(explain what a decision-tree is)

input $x_1, \dots, x_n$

output $f(x_1 \dots x_n)$

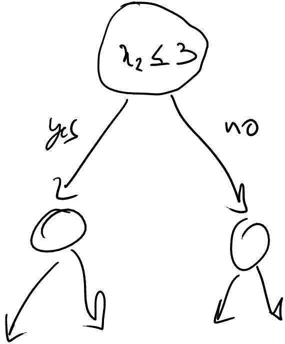


a decision-tree computes f if, for every leaf, $f(x)$ is the same for every input that would follow the path from root to that leaf

Previous results: the learning problem

(data space, queries, access to samples $(x, f(x))$, the goal is to produce a classifier)

x_1	x_2	x_3		obs
7	5	−2	...	0
				1
				⋮
				0
				0
				1
				⋮



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- In our setting, we assume that we are given the *full description* of a total function $f : \{0,1\}^n \to \{0,1\}$, either as a truth-table, or succinctly as a Boolean circuit, and we wish to find a decision-tree for computing f that makes as few queries as possible. I.e.:

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    - Our model is decision-trees.
    - Our complexity measure is decision-tree depth (also known as *query complexity*).
    - The given computational problem is a total Boolean function f,
    - which is either given as a truth-table, or as a circuit (we wish to understand both scenarios).

Our results

Theorem

The problem of computing the query complexity of f, when given the truth-table of f, is NC_1*-hard, and it can be computed by circuits of depth* $O(\log n \log\log n)$.

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The problem of computing the query complexity of f, when given a Boolean circuit for computing f, is PSPACE-complete.

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The first result is proven by the same technique.

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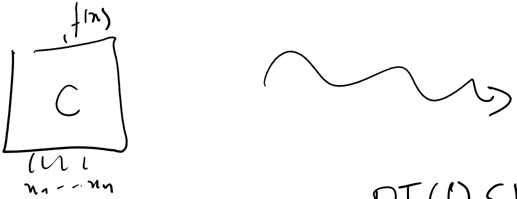
(DT$(f) \leq 0$ *iff f is constant, how about* DT$(f) \leq k$*?*)



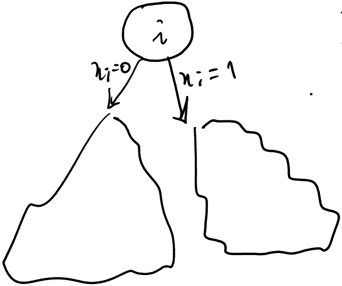
$\mathrm{DT}(f) = ?$

$$\mathrm{DT}(f) \leq k \iff$$

$$\exists_{i \in [n]} \; \mathrm{DT}(f|_{x_i=0}) \leq k-1$$

$$\wedge \; \mathrm{DT}(f|_{x_i=1}) \leq k-1$$

$$\iff \exists_{i \in [n]} \forall_{b \in \{0,1\}} \mathrm{DT}(f|_{x_i=b}) \leq k-1$$



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(DT$(f) \leq 0$ *iff* f *is constant, how about* DT$(f) \leq k$?)

The key difficulty: circuit-DT vs TQBF

(def. of TQBF, $\exists y_1 \forall x_1 \ldots$, TQBF as a two-player game vs circuit-DT as a two-player game)

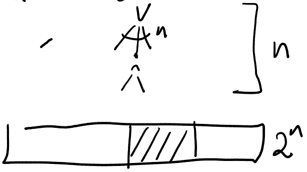
<u>TQBF</u>

input: $C(y_1, x_1, \ldots, y_n, x_n)$ a Boolean circuit

<u>output</u> whether or not

$$\exists y_1 \in \{0,1\} \; \forall x_1 \; \exists y_2 \; \forall x_2 \; \cdots \; \exists y_n \; \forall x_n \quad C(y_1, x_1, \ldots) = 1$$

vs Circuit-DT $(DT(f) \leq k)$



<u>input</u> $C(z_1, \ldots, z_n)$

<u>output</u>

$$\exists_{i_1 \in [n]} \; \forall z_{i_1} \; \exists_{i_2} \; \forall z_{i_2} \; \cdots \; \exists_{i_k} \; \forall z_k \quad C|_{z_{i_1} \cdots z_{i_k}} \text{ constant}$$

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- I will not prove the full theorem, but I will prove an important auxiliary lemma.
- This will give some idea of how one forces a circuit-DT game to behave like a TQBF game.

Consider the following function:

$$F_n(y_1, y_1', x_1, x_1', \ldots y_n, y_n', x_n, x_n') := f_1 \oplus g_1 \oplus \ldots \oplus f_n \oplus g_n,$$

where:

$$f_i = y_i \wedge y_i'$$

$$g_i = \begin{cases} x_i & \text{if } \overbrace{f_1 \oplus g_1 \oplus \ldots \oplus g_{i-1} \oplus f_i} = 1 \\ x_i' & \text{otherwise.} \end{cases}$$

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- How might Alice and Bob play the circuit-DT game on F_n?

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- How might Alice and Bob play the circuit-DT game on F_n?
- Remember, in the circuit-DT game, Alice chooses a variable, and Bob sets that variable to some value.
- Bob's goal is to make the game last as long as possible before the function becomes constant.

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- If Bob sets the chosen y variable to $\cancel{1}$ 0, then there is no need to check the other variable to know that $f_1 = 0$.

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- Suppose Alice chooses either y_1 or y'_1.
- If Bob sets the chosen y variable to 1, then there is no need to check the other variable to know that $f_1 = 0$. So Alice didn't need to ask about the other y variable, and she has saved one query.

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- By some other part of the construction which I will soon sketch, Bob will be forced to answer 0 to the second query.
- So by asking variables in the right order, Alice can force Bob to set y_1 to 1 or 0, as she desires.
- This is exactly the power that Alice has in the TQBF game.

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- Let me briefly show you the final construction.

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For the full text:

`https://eccc.weizmann.ac.il/report/2024/034/`

Thank you!