Cellular Automata: Communication Matters

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Overview

- ➜ Cellular Automata
- ➜ The Impact of the Number of Communications
- ➜ Decidability
- ➜ Limiting the Inter-Cell Bandwidth
- ➜ Cellular Automata with Minimal Communication

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Cellular Automata

General Idea

- Cellular automata are considered as homogeneously structured, massively parallel computing systems.
- Instances of problems to be solved can be encoded as strings with a finite number of different symbols.
- The input data supplied to CAs are strings of symbols.
- The output can be encoded in binary.

Cellular Automata

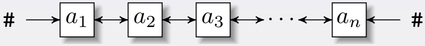
General Idea

- The computation can be decomposed into parallel processes, one for each bit of the output.
- Each process computes a function mapping the input to YES or NO.
- Given some set of input symbols, the set of all strings that are evaluated to YES is denoted by $L(\mathrm{DEVICE})$.
- $L(\mathrm{DEVICE})$ is called a formal language.

Cellular Automata

Two-Way (CA)

$$M = \langle S, F, A, B, \#, b_l, b_r, \delta \rangle$$



Set of messages: B (where $\perp$ means nothing to send)

Communication functions: $b_l, b_r \colon (S \cup \{\#\}) \to (B \cup \{\perp\})$

Local transition function: $\delta \colon (B \cup \{\perp\}) \times S \times (B \cup \{\perp\}) \to S$

Cellular Automata

One-Way (OCA)

$M = \langle S, F, A, B, \#, b_l, \delta \rangle$

$\boxed{a_1} \leftarrow \boxed{a_2} \leftarrow \boxed{a_3} \leftarrow \cdots \leftarrow \boxed{a_n} \leftarrow \#$

Set of messages: B (where $\perp$ means nothing to send)

Communication function: $b_l \colon (S \cup \{\#\}) \to (B \cup \{\perp\})$

Local transition function: $\delta \colon S \times (B \cup \{\perp\}) \to S$

Cellular Automata

Language Acceptor

$$M = \langle S, F, A, B, \#, b_l, \delta \rangle$$

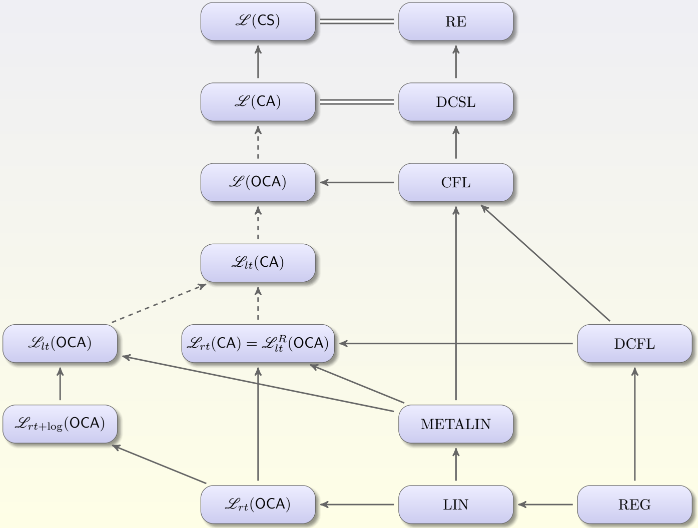
$$\boxed{a_1} \leftarrow \boxed{a_2} \leftarrow \boxed{a_3} \leftarrow \cdots \leftarrow \boxed{a_n} \leftarrow \#$$

A cellular language acceptor M evaluates the input string to YES or NO.

- The set of all strings that are evaluated to YES is denoted by $L(M)$.
- An input w is accepted (evaluated to YES), if the leftmost cell enters an accepting state at some step during the computation on w.
- For a mapping $t\colon \mathbb{N}_+ \to \mathbb{N}_+$, a formal language L is said to be of time complexity t, if all inputs w in L are accepted within at most $t(|w|)$ time steps.

Cellular Automata

Computational Capacity with Unlimited Communication



Example

Real-time OCA accepting $\{\, a^n b^n \mid n \geq 1 \,\}$

t	1	2	3	4	5	6	7	8	9	10	11	12	13
0	a	a	a	a	a	a	b	b	b	b	b	b	#
1	a	a	a	a	a	a	b	b	b	b	b	s	
2	a	a	a	a	a	c	b	b	b	b	s		
3	a	a	a	a	a	b	b	b	b	s			
4	a	a	a	a	c	b	b	b	s				
5	a	a	a	a	b	b	b	s					
6	a	a	a	c	b	b	s						
7	a	a	a	b	b	s							
8	a	a	c	b	s								
9	a	a	b	s									
10	a	c	s										
11	a	s											
12	$+$												

Problems and Questions

- How much communication is necessary for a computation?
- How can the cooperation of the cells be organized optimally?
- From the viewpoint of energy and the costs of communication links, it would be desirable to communicate a minimal number of times with a minimal bandwidth of the links.

The Impact of the Number of Communications

Limiting the Number of Messages

Number of communications between cell i and cell $i+1$ up to time t:

$$\mathrm{com}(i,t) = |\{\, j \mid 0 \leq j < t \text{ and } (b_r(c_j(i)) \neq \perp \text{ or } b_l(c_j(i+1)) \neq \perp)\,\}|$$

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Maximal number of communications between two cells:

$$\mathrm{mcom}(w) = \max\{\, \mathrm{com}(i, t(|w|)) \mid 1 \le i \le |w|-1 \,\}$$

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$$\mathrm{mcom}(w) = \max\{\, \mathrm{com}(i, t(|w|)) \mid 1 \leq i \leq |w|-1 \,\}$$

Total number of communications:

$$\mathrm{scom}(w) = \sum_{i=1}^{|w|-1} \mathrm{com}(i, t(|w|))$$

The Impact of the Number of Communications

Computational Capacity

Example

➜ $\{\, a^n b^n \mid n \geq 1 \,\} \in \mathscr{L}_{rt}(\text{MC}(\mathit{const})\text{-OCA})$

t	1	2	3	4	5	6	7	8	9	10	11	12	13
0	a	a	a	a	a	a	b	b	b	b	b	b	#
1	a	a	a	a	a	a	b	b	b	b	b	s	
2	a	a	a	a	a	c	b	b	b	b	s		
3	a	a	a	a	a	b	b	b	b	s			
4	a	a	a	a	c	b	b	b	s				
5	a	a	a	a	b	b	b	s					
6	a	a	a	c	b	b	s						
7	a	a	a	b	b	s							
8	a	a	c	b	s								
9	a	a	b	s									
10	a	c	s										
11	a	s											
12	+												

The Impact of the Number of Communications

Computational Capacity

Example

- $\{\, a^n b^n \mid n \geq 1 \,\} \in \mathscr{L}_{rt}(\mathrm{MC}(const)\text{-}\mathrm{OCA})$
- $\{\, a_1^n a_2^n \cdots a_k^n \mid n \geq 1 \,\} \in \mathscr{L}_{rt}(\mathrm{MC}(const)\text{-}\mathrm{OCA})$

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- $\{\, a^n b^m c^n d^m \mid n, m \geq 1 \,\} \in \mathscr{L}_{rt}(\mathrm{MC}(\mathit{const})\text{-OCA})$

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- $\{\, a^n b^m c^n d^m \mid n, m \geq 1 \,\} \in \mathscr{L}_{rt}(\mathsf{MC}(\mathit{const})\text{-}\mathsf{OCA})$
- $\{\, a^n b^{n_1} c^m b^{n_2} \mid n, m \geq 1 \wedge n_1, n_2 \geq 0 \wedge n_1 + n_2 = n \,\} \in \mathscr{L}_{rt}(\mathsf{MC}(\mathit{const})\text{-}\mathsf{OCA})$

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Example

- $\{\, a^n b^n \mid n \geq 1 \,\} \in \mathscr{L}_{rt}(\mathrm{MC}(\mathit{const})\text{-OCA})$
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- $\{\, a^n b^{n+\lfloor \sqrt{n} \rfloor} \mid n \geq 1 \,\} \in \mathscr{L}_{rt}(\mathrm{MC}(\mathit{const})\text{-OCA})$

Signal with slope $2n + \lfloor \sqrt{n} \rfloor$

The Impact of the Number of Communications

Computational Capacity

Theorem **[Vollmar 1981/1982]**

1. $\mathscr{L}_{rt}(\mathsf{MC}(const)\text{-}\mathsf{CA}) \subset \mathscr{L}_{rt}(\mathsf{SC}(n)\text{-}\mathsf{CA})$
2. $\mathsf{REG} \subset \mathscr{L}_{rt}(\mathsf{MC}(const)\text{-}\mathsf{CA}) \subset \mathscr{L}_{rt}(\mathsf{MC}(\sqrt{n})\text{-}\mathsf{CA}) \subset \mathscr{L}_{rt}(\mathsf{MC}(n)\text{-}\mathsf{CA})$
3. $\mathscr{L}_{lt}(\mathsf{MC}(const)\text{-}\mathsf{CA}) \subset \mathsf{NL}$

The Impact of the Number of Communications

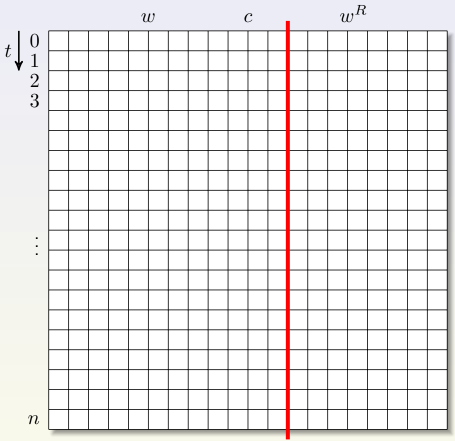
Towards an Infinite Communication Hierarchy

Theorem

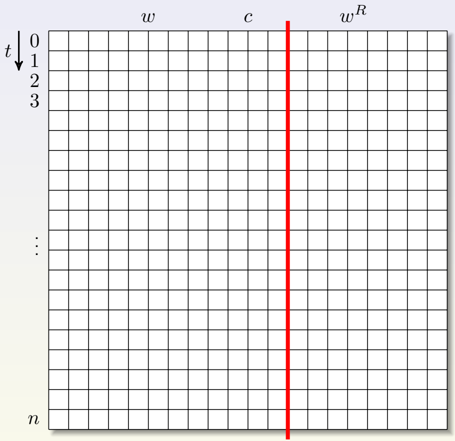
Let $f : \mathbb{N} \to \mathbb{N}$ be a function. If $f \in o(n^2/\log(n))$, then language

$$\{\, wcw^R \mid w \in \{a, b\}^+ \,\}$$

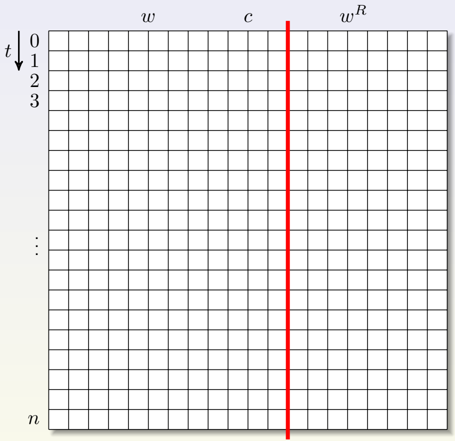
is not accepted by any real-time SC(f)-CA.



w
c
w^R
t
0
1
2
3
$\vdots$
n

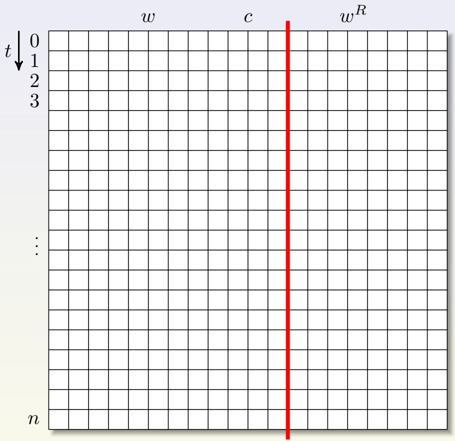


Messages: $r \in o(|w|/\log(|w|))$



Messages: $r \in o(|w|/\log(|w|))$

Possibilities: $(|B|+1)^2-1$



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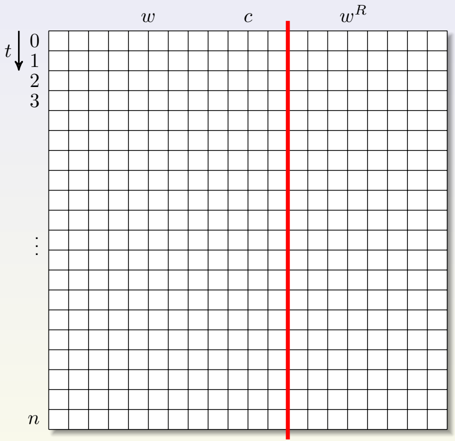
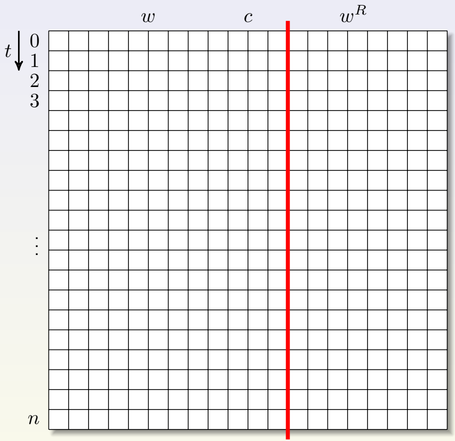
Possibilities: $((|B|+1)^2-1)^r$



Messages: $r \in o(|w|/\log(|w|))$

Possibilities: $\binom{n}{r}((|B|+1)^2-1)^r$



Messages: $r \in o(|w|/\log(|w|))$

Possibilities: $\binom{n}{r}((|B|+1)^2-1)^r \leq 2^{k_0 \log(n) r} \leq \sqrt{2^{|w|}}$

The Impact of the Number of Communications

Witness Languages for an Infinite Communication Hierarchy

$$\varphi_i(n) \quad = \quad \begin{cases} 2^n & \text{if } i = 1 \\ 2^{\varphi_{i-1}(n)} & \text{if } i \geq 2 \end{cases}$$

$$L_i \;=\; \{\, w\$^{\varphi_i(|w|)-2|w|}w^R \;\mid\; w \;\in\; \{a,b\}^+ \,\}$$

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Theorem

Let $i \geq 1$ be an integer and $f : \mathbb{N} \to \mathbb{N}$ be a function.

1. If $f \in o((n \log^{[i]}(n))/\log^{[i+1]}(n))$, then language L_i is not accepted by any real-time SC(f)-CA.

The Impact of the Number of Communications

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1. If $f \in o((n \log^{[i]}(n))/\log^{[i+1]}(n))$, then language L_i is not accepted by any real-time SC(f)-CA.
2. Language L_i is accepted by some real-time SC($n \log^{[i]}(n)$)-CA.

The Impact of the Number of Communications

Infinite Communication Hierarchy

$$n \log^{[i+1]}(n) \quad \in \quad o((n \log^{[i]}(n))/\log^{[i+1]}(n)).$$

Theorem

Let $i \geq 0$ be an integer. Then

$\mathscr{L}_{rt}(\mathrm{SC}(n \log^{[i+1]}(n))\text{-CA})$ is strictly included in $\mathscr{L}_{rt}(\mathrm{SC}(n \log^{[i]}(n))\text{-CA})$.

Decidability Problems

- Undecidability for real-time MC($const$)-OCAs by reduction of Hilbert's tenth problem.
- The problem is to decide for a given polynomial $p(x_1, x_2, \ldots, x_n)$ with integer coefficients whether there are integers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $p(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0$.

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- Construction of a (rather technical) language $L(p)$ which is empty if and only if $p(x_1, x_2, \ldots, x_n) = 0$ has no solution.

Lemma

The language $L(p)$ is accepted by some real-time $\mathrm{MC}(const)$-OCA.

Decidability Problems

Classical Problems

Lemma

The language $L(p)$ is accepted by some real-time $\mathrm{MC}(const)$-OCA.

Corollary

The problems emptiness, finiteness, infiniteness, universality, equivalence, inclusion, regularity, and context-freeness are undecidable for arbitrary real-time $\mathrm{MC}(const)$-OCAs.

Decidability Problems

Question: Are the communication restrictions themselves decidable?

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Answer: No.

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Answer: No.

- Given some real-time MC($const$)-OCA M', we consider the language $L_{M'} = \{\, a^{|w|}w \mid w \in L(M') \,\}$.
- $L_{M'}$ is accepted by a real-time OCA M, so that
- M is an MC($const$)-OCA if and only if $L_{M'}$ is finite.

Decidability Problems

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- $L_{M'}$ is accepted by a real-time OCA M, so that
- M is an $\mathrm{MC}(const)$-OCA if and only if $L_{M'}$ is finite.

Theorem

It is undecidable for an arbitrary real-time OCA whether it is a real-time $\mathrm{MC}(const)$-OCA.

Limiting the Inter-Cell Bandwidth

Difficulties

For communication-restricted devices, in general, one cannot utilize algorithmic design techniques such as

- ➜ speeding up the computation time beyond real time,

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- folding and unfolding space-time diagrams, and

Limiting the Inter-Cell Bandwidth

Difficulties

For communication-restricted devices, in general, one cannot utilize algorithmic design techniques such as

- speeding up the computation time beyond real time,
- data packing and unpacking,
- folding and unfolding space-time diagrams, and
- multi-layer programming.

Limiting the Inter-Cell Bandwidth

What Communication-Restricted Devices Can do

- There is a time-optimal one-bit solution of the famous Firing Squad Synchronization Problem [Mazoyer 1989].

Limiting the Inter-Cell Bandwidth

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- This implies:

$$\{ a^{2^n} \mid n \geq 0 \} \in \mathscr{L}_{rt}(\mathrm{CA}_1)$$

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- This implies:

$$\{\, a^{2^n} \mid n \geq 0 \,\} \in \mathscr{L}_{rt}(\mathsf{CA}_1)$$

$$\{\, a^{n^2} \mid n \geq 0 \,\} \in \mathscr{L}_{rt}(\mathsf{CA}_1)$$

$$\{\, a^p \mid p \text{ is prim} \,\} \in \mathscr{L}_{rt}(\mathsf{CA}_1)$$

$$\{\, a^p \mid p \text{ is a Fibonacci number} \,\} \in \mathscr{L}_{rt}(\mathsf{CA}_1)$$

Limiting the Inter-Cell Bandwidth

What Communication-Restricted Devices Cannot do

Theorem

For all $k \geq 1$, there is a regular language which is not accepted by any real-time CA$_k$.

Witness languages:

$$L_k = \{xvx \mid v \in \{a\}^* \text{ and } x \in \{a_0, \ldots, a_{2^{2k}}\}\}$$

Limiting the Inter-Cell Bandwidth

What Communication-Restricted Devices Cannot do

Theorem

For all $k \geq 1$, there is a regular language which is not accepted by any real-time CA_k.

Theorem

For all $k \geq 1$, $\mathscr{L}_{rt}(\mathrm{CA}_k) \subset \mathscr{L}_{rt}(\mathrm{CA}_{k+1})$.

Limiting the Inter-Cell Bandwidth

What Communication-Restricted Devices Cannot do

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Theorem

For all $k, r \geq 1$, $\mathscr{L}_{rt+r}(\mathrm{CA}_k) \subset \mathscr{L}_{rt+r+1}(\mathrm{CA}_k)$.

Limiting the Inter-Cell Bandwidth

What Communication-Restricted Devices Cannot do

$$
\begin{tikzcd}
\mathscr{L}_{lt}(\mathsf{CA}) \arrow[r, equal] & \mathscr{L}_{lt}(\mathsf{CA}_k) \\
\mathscr{L}_{lt}(\mathsf{OCA}) \arrow[u, dashed] \arrow[r, equal] & \mathscr{L}_{lt}(\mathsf{OCA}_k) \arrow[u, dashed] \\
\mathscr{L}_{rt}(\mathsf{CA})^R \arrow[u, equal] & \mathscr{L}_{rt}(\mathsf{CA}_k)^R \arrow[l] \arrow[u] \\
\mathscr{L}_{rt}(\mathsf{OCA}) \arrow[u] & \mathscr{L}_{rt}(\mathsf{OCA}_k) \arrow[l] \arrow[u] \\
\mathrm{REG} \arrow[u] &
\end{tikzcd}
$$

Cellular Automata with Minimal Communication

Question: What is the status of decidability problems when the communication of cellular automata is drastically reduced to its minimum, but kept still enough to have non-trivial devices?

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To identify the minimum of communication,

1. each two adjacent cells are allowed to communicate constantly often only,
2. only one type of message is provided,
3. the information flow is one-way, and
4. the time complexity is bounded to real time.

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To identify the minimum of communication,

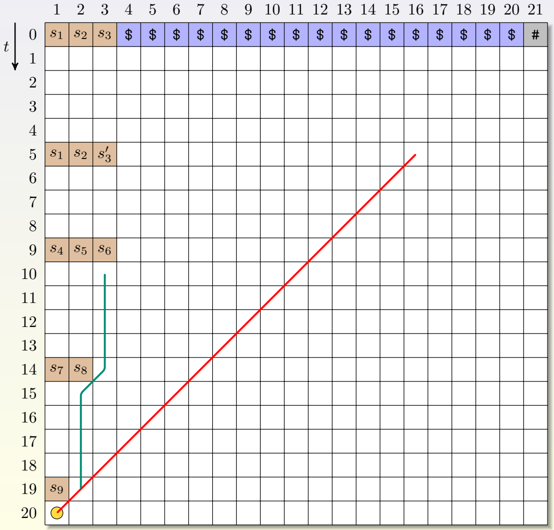
1. each two adjacent cells are allowed to communicate constantly often only,
2. only one type of message is provided,
3. the information flow is one-way, and
4. the time complexity is bounded to real time.
5. That is, we consider the class of real-time $\mathrm{MC}(const)$-OCA_1.
6. The computations are non-trivial.

Lemma

Let $M = \langle S, F, A, B, \#, b_l, \delta \rangle$ be a real-time MC($const$)-OCA and $\$ \notin A$ be a new symbol. Then a real-time MC($const$)-OCA$_1$ M' accepting the language $\{\, w\$^{(|B|+2)(|w|+1)}v \mid v \in \{\$, A(A \cup \{\$\})^*\}, w \in L(M)\,\}$ can effectively be constructed.

Cellular Automata with Minimal Communication

One-way One-bit O(One)-message Cellular Automata



Cellular Automata with Minimal Communication

Decidability

Theorem

Emptiness, finiteness, infiniteness, equivalence, inclusion, regularity, and context-freeness are undecidable for real-time $\mathrm{SC}(n)$-OCA_1 and $\mathrm{MC}(const)$-OCA_1.

Cellular Automata with Minimal Communication

Decidability

Theorem

Emptiness, finiteness, infiniteness, equivalence, inclusion, regularity, and context-freeness are undecidable for real-time $\mathrm{SC}(n)$-OCA_1 and $\mathrm{MC}(\mathit{const})$-$\mathrm{OCA}_1$.

Emptiness, finiteness, infiniteness, equivalence, inclusion, regularity, and context-freeness are undecidable for real-time $\mathrm{SC}(n)$-OCA_2 and $\mathrm{MC}(\log n)$-OCA_2 accepting bounded languages.

Cellular Automata with Minimal Communication

Open Problems

Open Problems

How about the decidability for real-time $\mathrm{SC}(n)$-OCA_1 or $\mathrm{MC}(o(\log n))$-OCA_1 accepting bounded languages?

Are there other natural restrictions that yield decidable properties?