Synchronised Shuffles and Team-like Automata

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Joint work with Sabine Broda, Rogério Reis, and António Machiavelo.

$$ab \sqcup\!\sqcup\, bc = \{abbc, abcb, bacb, babc, bcab\}$$

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$x, x' \in \Sigma^\star$ and $a, a' \in \Sigma$

$$\begin{aligned} x \sqcup\!\sqcup\, \varepsilon = \varepsilon \sqcup\!\sqcup\, x &= \{x\} \\ ax \sqcup\!\sqcup\, a'x' &= \{\, az \mid z \in x \sqcup\!\sqcup\, a'x' \,\} \cup \{\, a'z \mid z \in ax \sqcup\!\sqcup\, x' \,\}. \end{aligned}$$

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$$L_1 \sqcup\!\sqcup\, L_2 = \bigcup_{x \in L_1, y \in L_2} x \sqcup\!\sqcup\, y$$

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$$L_1 \sqcup\!\sqcup\, L_2 = \bigcup_{x \in L_1, y \in L_2} x \sqcup\!\sqcup\, y$$

$\sqcup\!\sqcup$ is commutative, associative, and it distributes over union If $L_1, L_2 \subseteq \Sigma^\star$ are regular, then $L_1 \sqcup\!\sqcup\, L_2$ is regular.

- formal languages [GS65, RS97]
    - expressivity
    - closures
    - algebraic caracterisations
    - decompositions
    - complexity
- group theory: in the context of abelian groups and free groups [EL53, CFL58]
- concurrency theory [ORR78, BPS01]
- molecular biology [tBMVM05]
- and card games, of course

Our conclusion is, in brief, that shuffling is complex.

Ogden, Riddle, and Round [ORR78]

It still remains a mysterious operation on regular languages.

Berstel, Boasson, Carton, Pin, and Restivo [BBC+10]

Despite the length of time since the operator was introduced there remains a number of standard formal language theoretic questions involving shuffle that are unsolved.

Bieger, Daley, and McQuillan [BM14]

$$L_i = \mathcal{L}(A_i), \;\; n_i = |A_i|, \; i = 1, 2$$

$$\mathsf{isc}(L_1 \sqcup\!\sqcup L_2) = 2^{n_1 n_2} - 1$$

[CSY02]

$$\mathsf{sc}(L_1 \sqcup\!\sqcup L_2) \leq 2^{n_1 n_2 - 1} + 2^{(n_1-1)(n_2-1)}(2^{n_1-1} - 1)(2^{n_2-1} - 1)$$

[BJL+16, CLP20]

$$\mathsf{sc}(u \sqcup\!\sqcup v) \in O(2^{n_2-1}(n_1 - 2)) \;\; (n_1 \leq n_2)$$

[BDM10]

$$\Sigma = \{\sigma_1, \ldots, \sigma_m\}$$

$$\alpha \to \varepsilon \mid \emptyset \mid \sigma_1 \mid \cdots \mid \sigma_m \mid (\alpha + \alpha) \mid (\alpha\alpha) \mid (\alpha^\star)$$

$$\varepsilon(\alpha) = \begin{cases} \varepsilon & \text{if } \varepsilon \in \mathcal{L}(\alpha) \\ \emptyset & \text{otherwise} \end{cases}$$

$$\alpha \doteq \alpha' \iff \mathcal{L}(\alpha) = \mathcal{L}(\alpha')$$

We denote,

$$\mathrm{RE} = \mathrm{RE}(+, \cdot, \star)$$

Position based NFAs:

$\mathcal{A}_{\mathcal{G}\ell}$ Position/Glushkov [Glu61]

$\mathcal{A}_{\mathrm{F}}$ Follow [IY03]

Derivative based NFAs

$\mathcal{A}_{\mathrm{PD}}$ Partial Derivatives [Ant96]

$\mathcal{A}_{\mathrm{Pre}}$ Prefix [Yam14]

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$\mathcal{A}_{\mathrm{PD}}$ Partial Derivatives [Ant96]

$\mathcal{A}_{\mathrm{PD}}(\alpha) \simeq \mathcal{A}_{\mathcal{G}\ell}(\alpha)/_{\equiv_c}$ [CZ02]

$\mathcal{A}_{\mathrm{Pre}}$ Prefix [Yam14]

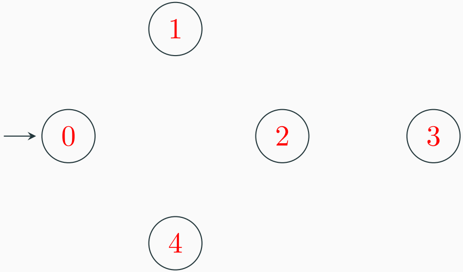
$\mathcal{A}_{\mathrm{Pre}}(\alpha) \simeq \mathcal{A}_{\mathcal{G}\ell}(\alpha)/_{\equiv_p}$ [BMMR21b]

$$\alpha = (b+ab)^\star + b^\star$$

$\mathcal{A}_{\mathscr{G}\ell}((b + ab)^\star + b^\star)$

$$\overline{\alpha} = (b_1 + a_2 b_3)^\star + b_4^\star$$

$$\mathsf{Pos}(\overline{\alpha}) = \{1, 2, 3, 4\}$$



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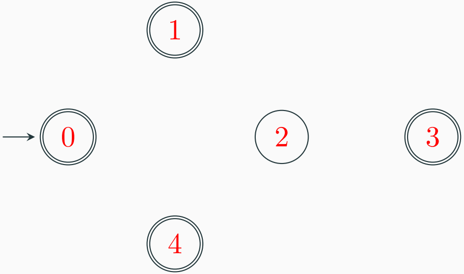
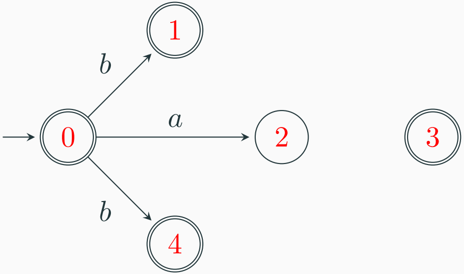


$$\mathsf{Last}_0(\overline{\alpha}) = \{0, 1, 3, 4\}$$

$$\overline{\alpha} = (b_1 + a_2 b_3)^\star + b_4^\star$$



$$\mathsf{First}(\overline{\alpha}) = \{1, 2, 4\}$$

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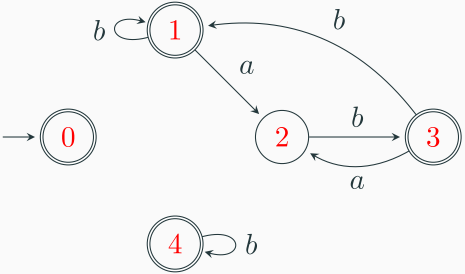
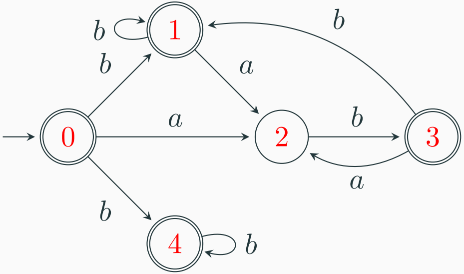


$$\text{Follow}(\overline{\alpha}) = \{(1,1),(1,2),(2,3),(3,1),(3,2),(4,4)\}$$

$$\alpha = (b + ab)^\star + b^\star$$



$$\text{Select}(\text{Follow}(\alpha, 1), b) = \text{Select}(\{1, 2\}, b) = \{1\}$$

Position/Glushkov Automaton, $\mathcal{A}_{\mathcal{G}\ell}(\alpha)$

$$\mathcal{A}_{\mathcal{G}\ell}(\alpha) = \langle \mathsf{Pos}_0, \Sigma, \delta_{\mathcal{G}\ell}, 0, \mathsf{Last}_0(\alpha) \rangle$$

$$\mathsf{Pos}_0(\alpha) = \mathsf{Pos}(\alpha) \cup \{0\}$$

$$\mathsf{Last}_0(\alpha) = \mathsf{Last}(\alpha) \cup \varepsilon(\alpha)\{0\}$$

$$\mathsf{Follow}(\alpha, 0) = \mathsf{First}(\alpha)$$

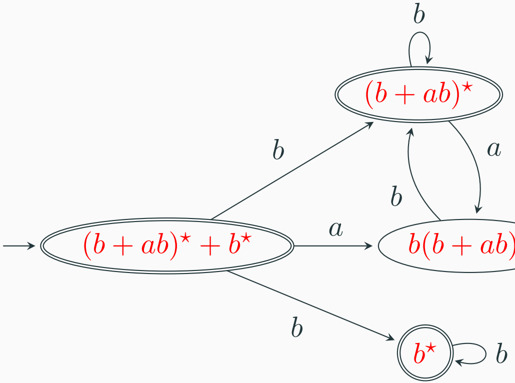
$$\delta_{\mathcal{G}\ell}(i, \sigma) = \mathsf{Select}(\mathsf{Follow}(\overline{\alpha}, i), \sigma)$$

$$|\mathcal{A}_{\mathcal{G}\ell}(\alpha)|_Q = |\alpha|_\Sigma + 1 = n$$

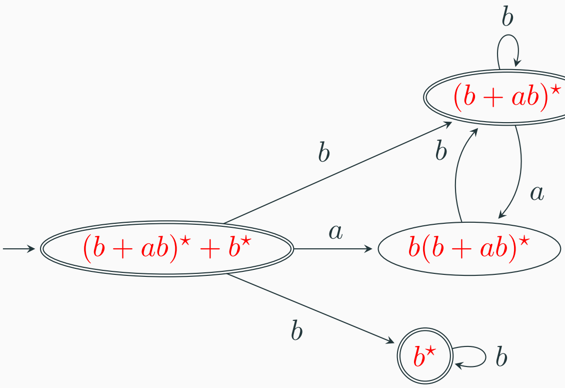
$$|\mathcal{A}_{\mathcal{G}\ell}(\alpha)|_\delta = \Theta(n^2)$$

Partial Derivatives for RE, $\partial_w(\alpha)$

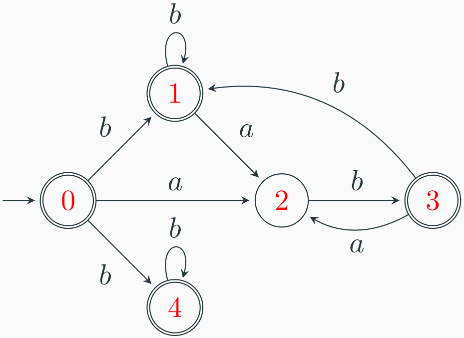
$$
\begin{aligned}
\mathcal{L}(\partial_w(\alpha)) &= \{w' \mid ww' \in \mathcal{L}(\alpha)\} \\
\partial_\sigma(\emptyset) &= \partial_\sigma(\varepsilon) = \emptyset, \\
\partial_\sigma(\sigma') &= \begin{cases} \{\varepsilon\} & \text{if } \sigma' = \sigma \\ \emptyset & \text{otherwise,} \end{cases} \\
\partial_\sigma(\alpha + \alpha') &= \partial_\sigma(\alpha) \cup \partial_\sigma(\alpha'), \\
\partial_\sigma(\alpha\alpha') &= \partial_\sigma(\alpha)\alpha' \cup \varepsilon(\alpha)\partial_\sigma(\alpha'), \\
\partial_\sigma(\alpha^\star) &= \partial_\sigma(\alpha)\alpha^\star \\
\partial_\varepsilon(\alpha) &= \{\alpha\} \\
\partial_{w\sigma}(\alpha) &= \bigcup_{\beta \in \partial_\sigma(\alpha)} \partial_w(\beta) \\
\mathsf{PD}(\alpha) &= \bigcup_{w \in \Sigma^\star} \partial_w(\alpha)
\end{aligned}
$$


b
$(b+ab)^\star$
b
a
b
a
$(b+ab)^\star + b^\star$
$b(b+ab)^\star$
b
$b^\star$
b

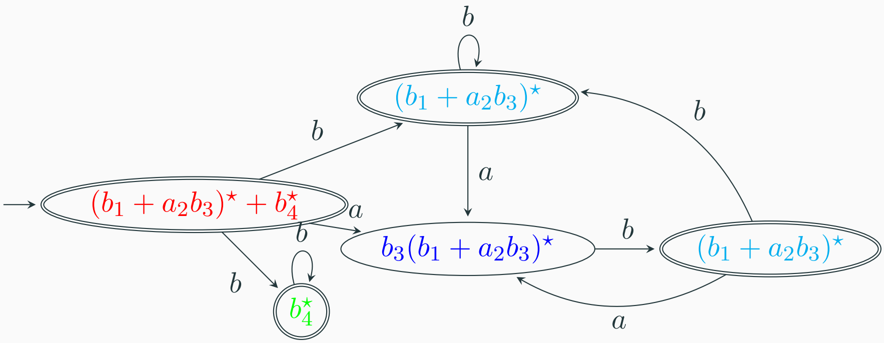
$\mathcal{A}_{\mathrm{PD}}((b+ab)^\star + b^\star)$



- $\mathcal{A}_{\mathrm{PD}}(\alpha) = \langle \mathsf{PD}(\alpha), \Sigma, \delta, \alpha, F \rangle$,
  - $F = \{\, \beta \in \mathsf{PD}(\alpha) \mid \varepsilon(\beta) = \varepsilon \,\}$,
  - $\delta(\beta, \sigma) = \partial_\sigma(\beta)$



- $\bigcup_{w \in \Sigma_{\overline{\alpha}}^{\star}} \partial_{w\sigma_i}(\overline{\alpha}) = \{c(\overline{\alpha}, i)\}$ if not empty
- $i \equiv_c j$ if $\overline{c(\overline{\alpha}, i)} = \overline{c(\overline{\alpha}, j)}$
- $\mathcal{A}_{\mathrm{PD}}(\alpha) \simeq \mathcal{A}_{\mathcal{G}\ell}(\alpha)/_{\equiv_c}$ [CZ02]



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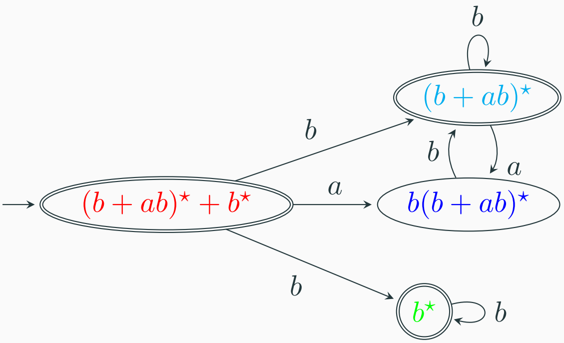

- $\bigcup_{w \in \Sigma^\star_{\overline{\alpha}}} \partial_{w\sigma_i}(\overline{\alpha}) = \{c(\overline{\alpha}, i)\}$ if not empty
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- $\mathcal{A}_{\mathrm{PD}}(\alpha) \simeq \mathcal{A}_{\mathscr{G}\!\ell}(\alpha)/_{\equiv_c}$ [CZ02]

Notation

$$
\begin{aligned}
\mathrm{RE}(\shuffle) &= \mathrm{RE}(+, \cdot, \star, \shuffle) \\
\mathrm{RE}(\cap) &= \mathrm{RE}(+, \cdot, \star, \cap) \\
\mathrm{RE}(\shuffle, \cap) &= \mathrm{RE}(+, \cdot, \star, \shuffle, \cap) \\
&\vdots
\end{aligned}
$$

- Membership for RE($\sqcup\!\sqcup$) is NP-complete [ORR78, MS94]
- Membership for RE($\cap$) is LOGCFL-complete [Pet02]
  (NL-complete for RE)
- Inequivalence for RE($\sqcup\!\sqcup, \cap$) is EXP-complete [MS94]
  (PSPACE-complete for RE)
- RE($\sqcup\!\sqcup, \cap$) $\Longrightarrow$ NFA exponential trade-off [MS94]
  (atmost quadratic for RE)
- RE($\sqcup\!\sqcup, \cap$) $\Longrightarrow$ RE double exponential trade-off [Gel10, GH09]
- RE($\sqcup\!\sqcup, \cap$) $\Longrightarrow$ DFA double exponential trade-off [Gel10]
  (exponential for RE)

Derivative Methods are easily extended to

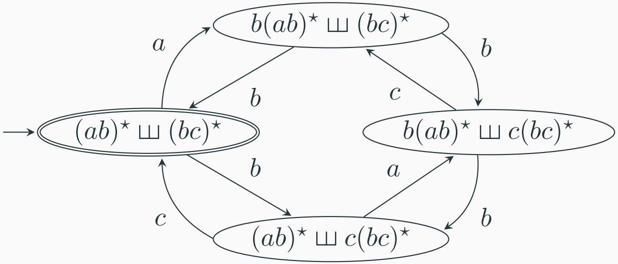
- Boolean Operations: $\cap$, $\neg$, etc
- Shuffle Operations:
  - interleave $\sqcup\!\sqcup$ [BMMR18]
  - several synchronised shuffles [ST19]

Position Methods extensions are not so easy

- operations compatible with positions: multi-bar and multi-tilde [CCM12]
- $\cap$ (*indexed sets*) [BMMR16]
- $\sqcup\!\sqcup$ using locations [BMMR21a]
- synchronised shuffles: using locations [BMMR23b]

$$\partial_\sigma(\alpha \sqcup\!\sqcup \beta) = \partial_\sigma(\alpha) \sqcup\!\sqcup \{\beta\} \cup \{\alpha\} \sqcup\!\sqcup \partial_\sigma(\beta)$$

$$\partial_\sigma(\alpha ⧢ \beta) = \partial_\sigma(\alpha) ⧢ \{\beta\} \cup \{\alpha\} ⧢ \partial_\sigma(\beta)$$



$$\alpha \quad = \quad (ab)^\star \sqcup\!\sqcup (bc)^\star$$

$$\overline{\alpha} \quad = \quad (a_1 b_2)^\star \sqcup\!\sqcup\, (b_3 c_4)^\star$$

$$\begin{aligned} \overline{\alpha} &= (a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star \\ \mathsf{Pos}(\alpha) &= \{1, 2, 3, 4\} \end{aligned}$$

$$\begin{aligned}
\overline{\alpha} &= (a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star \\
\mathsf{Pos}(\alpha) &= \{1, 2, 3, 4\} \\
\mathsf{First}(\alpha) &= ? \\
\mathsf{Last}(\alpha) &= ? \\
\mathsf{Follow}(\alpha) &= ?
\end{aligned}$$

$$(a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$$

$(a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$

a_1

$1, 0$

$(a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$

$a_1 b_3$

$1, 3$

$(a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$

$a_1 b_3 c_4$

$1, 4$

$(a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$

$a_1 b_3 c_4 b_2$

$2, 4$

$$(a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$$

$$a_1 b_3 c_4 b_2 \in \mathcal{L}(\overline{\alpha})$$

- In each step there are several active "follow" positions: sets of positions
- It is handy to know if those positions are from the left or the right component of the shuffle.

$$\begin{aligned}\overline{\alpha} &= (a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star \\ \mathsf{Pos}(\alpha) &= \{1, 2, 3, 4\}\end{aligned}$$

$$\begin{aligned}
\overline{\alpha} &= (a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star \\
\mathsf{Pos}(\alpha) &= \{1, 2, 3, 4\} \\
\mathsf{Loc}(\alpha) &= \{(1, 0), (0, 3), (0, 4), (2, 0), (1, 3), (1, 4), (2, 3), (2, 4)\}
\end{aligned}$$

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\mathsf{First}(\alpha) &= \{(a, (1, 0)), (b, (0, 3))\}
\end{aligned}
$$

$$
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\mathsf{First}(\alpha) &= \{(a, (1, 0)), (b, (0, 3))\} \\
\mathsf{Last}(\alpha) &= \{(2, 0), (0, 4), (2, 4)\} \\
\mathsf{Follow}(\alpha) &= \{((1, 0), b, (2, 0)), ((1, 0), b, (1, 3)), ((0, 3), a, (1, 3)), \\
&\quad (0, 3), c, (0, 4)), ((2, 0), a, (1, 0)), ((2, 0), b, (2, 3)), \\
&\quad ((0, 4), a, (1, 4)), ((1, 4), b, (1, 3)), ((1, 4), b, (2, 4)), \\
&\quad ((2, 3), a, (1, 3)), ((1, 3), c, (2, 4)), ((1, 3), b, (2, 3)), \ldots\}
\end{aligned}$$

$$\mathsf{Loc}(\varepsilon) = \emptyset,\ \mathsf{Loc}(\sigma_i) = \{i\},\ \mathsf{Loc}((\overline{\alpha})^\star) = \mathsf{Loc}(\overline{\alpha})$$

$$\mathsf{Loc}(\overline{\alpha}_1 + \overline{\alpha}_2) = \mathsf{Loc}(\overline{\alpha}_1\overline{\alpha}_2) = \mathsf{Loc}(\overline{\alpha}_1) \cup \mathsf{Loc}(\overline{\alpha}_2)$$

$$\mathsf{Loc}(\overline{\alpha}_1 \sqcup\!\sqcup \overline{\alpha}_2) = \mathsf{Loc}(\overline{\alpha}_1) \times \mathsf{Loc}(\overline{\alpha}_2) \cup \mathsf{Loc}(\overline{\alpha}_1) \times \{0\} \cup \{0\} \times \mathsf{Loc}(\overline{\alpha}_2)$$

$$\mathsf{Loc}(\varepsilon) = \emptyset,\ \mathsf{Loc}(\sigma_i) = \{i\},\ \mathsf{Loc}((\overline{\alpha})^\star) = \mathsf{Loc}(\overline{\alpha})$$

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$|\,\mathsf{Loc}(\alpha)| \leq 2^{|\alpha|_\Sigma} - 1$

$$\mathsf{First}(\sigma_i) = \{(\overline{\sigma_i}, i)\}$$

$$\mathsf{First}(\overline{\alpha}_1 \sqcup \overline{\alpha}_2) = \{(\sigma, (p, 0)) \mid (\sigma, p) \in \mathsf{First}(\overline{\alpha}_1)\}$$
$$\cup \{(\sigma, (0, p)) \mid (\sigma, p) \in \mathsf{First}(\overline{\alpha}_2)\}$$

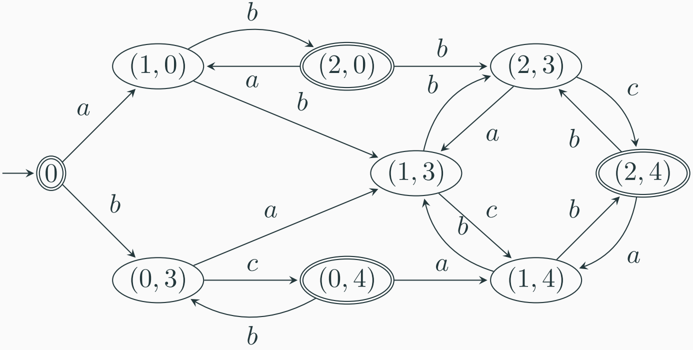
$$\begin{aligned}\mathsf{Last}(\overline{\alpha}_1 \sqcup\!\sqcup \overline{\alpha}_2) = {} & \mathsf{Last}(\overline{\alpha}_1) \times \mathsf{Last}(\overline{\alpha}_2) \\ & \cup\ \varepsilon(\overline{\alpha}_1)(\{0\} \times \mathsf{Last}(\overline{\alpha}_2)) \cup \varepsilon(\overline{\alpha}_2)(\mathsf{Last}(\overline{\alpha}_1) \times \{0\})\end{aligned}$$

$$\begin{aligned}\mathsf{Follow}(\overline{\alpha}_1 \sqcup \overline{\alpha}_2, (p_1, p_2)) = &\ \{\, (\sigma, (p_1', p_2)) \mid (\sigma, p_1') \in \mathsf{Follow}(\overline{\alpha}_1, p_1) \,\} \\ \cup &\ \{\, (\sigma, (p_1, p_2')) \mid (\sigma, p_2') \in \mathsf{Follow}(\overline{\alpha}_2, p_2) \,\}\end{aligned}$$

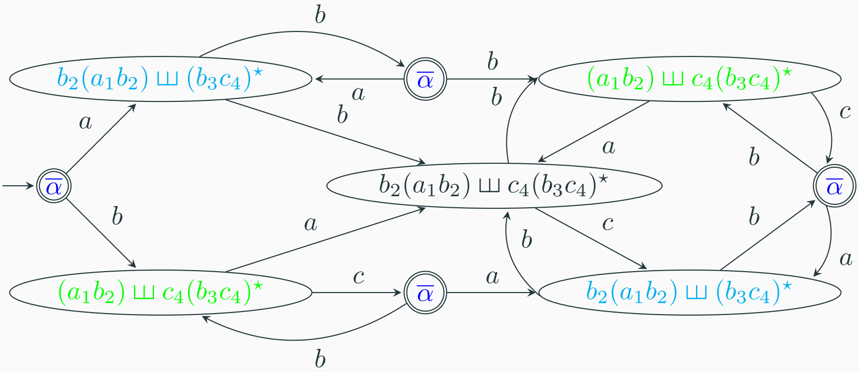
$$\mathsf{Loc}_0(\overline{\alpha}) = \{0, (0,3), (0,4), (1,0), (2,0), (1,3), (1,4), (2,3), (2,4)\}$$

$$\mathsf{First}(\overline{\alpha}) = \{(a, (1,0)), (b, (0,3))\}$$

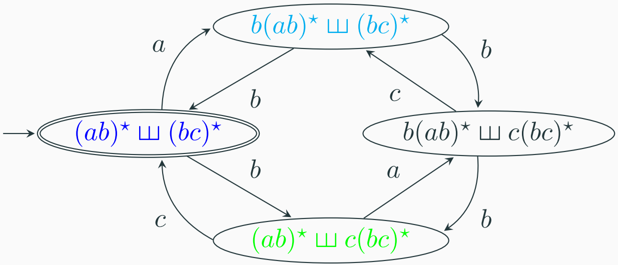
$$\mathsf{Last}_0(\overline{\alpha}) = \{0, (0,4), (2,0), (2,4)\}.$$



$$\overline{\alpha} = (a_1 b_2)^\star \sqcup\!\sqcup (b_3 c_4)^\star$$



$$\alpha = (ab)^\star \sqcup\!\sqcup (bc)^\star$$



$$p \equiv_c q \text{ if } \overline{\mathsf{c}(\overline{\alpha}, p)} = \overline{\mathsf{c}(\overline{\alpha}, q)}$$

- some letters must synchronise instead of interleave
- common letters must synchronise, studied first in [SS, dS84, LR99]
- M. ter Beek, J. Kleijn, and G. Rozenberg [tBK03, tBEKR03] within the framework of Team Automata considered that all occurrences of specific letters must be synchronised (*strong synchronised shuffle*).
- Further versions where studied by [tBMVM05, MMV06] in connection with biological models: *arbitrary synchronised shuffle*, *weak synchronised shuffle*, *synchronised shuffle on backbones*, ...

$$ab \overset{\mathsf{s}}{\|}_{\{b\}} bc = \{abc\}$$

$$ab \;{}^{\mathsf{s}}\|_{\{b\}}\; bc = \{abc\}$$

$$abca \;{}^{\mathsf{s}}\|_{\{a\}}\; ada = \{abcda, abdca, adbca\} = a(bc \sqcup d)a$$

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$\Gamma \subseteq \Sigma$, synchronising alphabet

$$u = u_1\sigma_1 \cdots \sigma_{n-1}u_n$$

$$v = v_1\sigma_1 \cdots \sigma_{n-1}v_n$$

with $\sigma_i \in \Gamma$ and $u_i, v_i \in (\Sigma \setminus \Gamma)^\star$,

$$u \overset{\mathsf{s}}{\|}_{\Gamma} v = (u_1 \sqcup\!\sqcup v_1)\sigma_1 \cdots \sigma_{n-1}(u_n \sqcup\!\sqcup v_n)$$

$$ab \;{}^{\mathsf{s}}\|_{\{b\}}\; bc = \{abc\}$$

$$abca \;{}^{\mathsf{s}}\|_{\{a\}}\; ada = \{abcda, abdca, adbca\} = a(bc \sqcup\!\sqcup d)a$$

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$$u = u_1\sigma_1 \cdots \sigma_{n-1}u_n$$

$$v = v_1\sigma_1 \cdots \sigma_{n-1}v_n$$

with $\sigma_i \in \Gamma$ and $u_i, v_i \in (\Sigma \setminus \Gamma)^\star$,

$$u \;{}^{\mathsf{s}}\|_{\Gamma}\; v = (u_1 \sqcup\!\sqcup v_1)\sigma_1 \cdots \sigma_{n-1}(u_n \sqcup\!\sqcup v_n)$$

Note: ${}^{\mathsf{s}}\|_{\emptyset} = \sqcup\!\sqcup$ and ${}^{\mathsf{s}}\|_{\Sigma} = \cap$

Letters in Γ can be synchronised or not

$$ab \;\mathsf{a}\|_{\{b\}}\; bc = ab \sqcup\!\sqcup\, bc \cup \{abc\}$$

Arbitrary Synchronised Shuffle

Letters in Γ can be synchronised or not

$$ab \,{}^{\mathsf{a}}\!\|_{\{b\}}\, bc = ab \sqcup\!\sqcup\, bc \cup \{abc\}$$

$\Gamma \subseteq \Sigma$, synchronising alphabet

$$u = u_1\sigma_1 \cdots \sigma_{n-1}u_n$$

$$v = v_1\sigma_1 \cdots \sigma_{n-1}v_n$$

with $\sigma_i \in \Gamma$ and $u_i, v_i \in \Sigma^\star$,

$$u \,{}^{\mathsf{a}}\!\|_\Gamma\, v = (u_1 \sqcup\!\sqcup\, v_1)\sigma_1 \cdots \sigma_{n-1}(u_n \sqcup\!\sqcup\, v_n)$$

$$\begin{aligned} acab \mathbin{|||_{ab}} bab &= (ac \sqcup\!\sqcup\, b)\{ab\} \cup (\varepsilon \sqcup\!\sqcup\, b)a(ca \sqcup\!\sqcup\, \varepsilon)b \\ &= \{acbab, abcab, bacab\} \cup \{bacab\} \\ &= \{acbab, abcab, bacab\} \end{aligned}$$

$$acab \mathbin{|||_{ab}} bab = (ac \sqcup\!\sqcup\, b)\{ab\} \cup (\varepsilon \sqcup\!\sqcup\, b)a(ca \sqcup\!\sqcup\, \varepsilon)b$$
$$= \{acbab, abcab, bacab\} \cup \{bacab\}$$
$$= \{acbab, abcab, bacab\}$$

$$u = u_1\sigma_1 \cdots \sigma_{n-1}u_n$$

$$v = v_1\sigma_1 \cdots \sigma_{n-1}v_n$$

with $u_i, v_i \in \Sigma^\star$ and

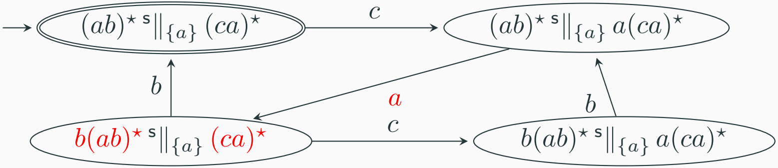
$$t = \sigma_1 \cdots \sigma_{n-1}$$

$$u \mathbin{|||_t} v = (u_1 \sqcup\!\sqcup\, v_1)\sigma_1 \cdots \sigma_{n-1}(u_n \sqcup\!\sqcup\, v_n)$$

Partial Derivatives for $\mathrm{RE}({}^{\mathsf{s}}\|_\Gamma)$

$\Gamma \subseteq \Sigma$, synchronising alphabet

$$\partial_\sigma(\alpha \,{}^{\mathsf{s}}\|_\Gamma\, \beta) = \begin{cases} \partial_\sigma(\alpha) \,{}^{\mathsf{s}}\|_\Gamma\, \partial_\sigma(\beta) \text{ if } \sigma \in \Gamma, \\ \partial_\sigma(\alpha) \,{}^{\mathsf{s}}\|_\Gamma\, \{\beta\} \cup \{\alpha\} \,{}^{\mathsf{s}}\|_\Gamma\, \partial_\sigma(\beta), \text{ otherwise.} \end{cases}$$


$(ab)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star$
c
$(ab)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, a(ca)^\star$
b
a
b
c
$b(ab)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, (ca)^\star$
$b(ab)^\star \,{}^{\mathsf{s}}\|_{\{a\}}\, a(ca)^\star$

$$\mathbb{B} = \{\top, \bot\},\ s_a = \{\langle \top, \top \rangle\} \text{ and } s_b = s_c = \{\langle \bot, \top \rangle, \langle \top, \bot \rangle\}$$

$$(a_1 b_2)^{\star\,\mathsf{s}}\|_{\{a\}}\,(c_3 a_4)^{\star}$$

$$\mathbb{B} = \{\top, \bot\},\ s_a = \{\langle\top, \top\rangle\} \text{ and } s_b = s_c = \{\langle\bot, \top\rangle, \langle\top, \bot\rangle\}$$

$$(a_1 b_2)^\star {}^{\mathsf{s}}\|_{\{a\}} (c_3 a_4)^\star$$

$$\mathsf{First}((a_1 b_2)^\star) = \{(a, 1)\}$$

$$\mathsf{First}((c_3 a_4)^\star) = \{(c, 3)\}$$

$$\mathsf{First}((a_1 b_2)^\star {}^{\mathsf{s}}\|_{\{a\}} (c_3 a_4)^\star) = \{(c, (0, 3))\}$$

$$\mathbb{B} = \{\top, \bot\},\ s_a = \{\langle\top, \top\rangle\} \text{ and } s_b = s_c = \{\langle\bot, \top\rangle, \langle\top, \bot\rangle\}$$

$$(a_1 b_2)^{\star}\ {}^{\mathsf{s}}\!\|_{\{a\}}\ (c_3 a_4)^{\star}$$

$$\mathsf{First}((a_1 b_2)^{\star}) = \{(a, 1)\}$$

$$\mathsf{First}((c_3 a_4)^{\star}) = \{(c, 3)\}$$

$$\mathsf{First}((a_1 b_2)^{\star}\ {}^{\mathsf{s}}\!\|_{\{a\}}\ (c_3 a_4)^{\star}) = \{(c, (0, 3))\}$$

$$\mathsf{First}(\overline{\alpha}_1\ {}^{\mathsf{s}}\!\|_{\Gamma}\ \overline{\alpha}_2) = \bigcup_{\substack{\sigma \in \Sigma \\ [\mathsf{b}_1, \mathsf{b}_2] \in s_\sigma}} \{(\sigma, (p_1, p_2)) \mid (\sigma, p_i) \in \mathsf{First}(\alpha_i) \text{ if } \mathsf{b}_i; p_i = 0 \text{ otherwise}\}$$

where $s_\sigma = \{\langle\top, \top\rangle\}$ if $\sigma \in \Gamma$; $s_\sigma = \{\langle\top, \bot\rangle, \langle\bot, \top\rangle\}$ otherwise.

$$\mathbb{B} = \{\top, \bot\},\ s_a = \{\langle\top, \top\rangle\} \text{ and } s_b = s_c = \{\langle\bot, \top\rangle, \langle\top, \bot\rangle\}$$

$$(a_1 b_2)^\star {}^{\mathsf{s}}\|_{\{a\}}\,(c_3 a_4)^\star$$

$$\mathsf{First}((a_1 b_2)^\star) = \{(a, 1)\} \quad \mathsf{Follow}((c_3 a_4)^\star, 3) = \{(a, 4)\}$$

$$\mathsf{Follow}((a_1 b_2)^\star {}^{\mathsf{s}}\|_{\{a\}}\,(c_3 a_4)^\star, (0, 3)) = \{(a, (1, 4))\}$$

$$\mathbb{B} = \{\top, \bot\},\ s_a = \{\langle \top, \top \rangle\} \text{ and } s_b = s_c = \{\langle \bot, \top \rangle, \langle \top, \bot \rangle\}$$

$$(a_1 b_2)^{\star}\ {}^{\mathsf{s}}\|_{\{a\}}\ (c_3 a_4)^{\star}$$

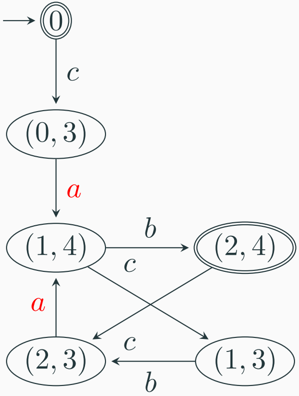
$$\mathsf{First}((a_1 b_2)^{\star}) = \{(a, 1)\} \quad \mathsf{Follow}((c_3 a_4)^{\star}, 3) = \{(a, 4)\}$$

$$\mathsf{Follow}((a_1 b_2)^{\star}\ {}^{\mathsf{s}}\|_{\{a\}}\ (c_3 a_4)^{\star}, (0, 3)) = \{(a, (1, 4))\}$$

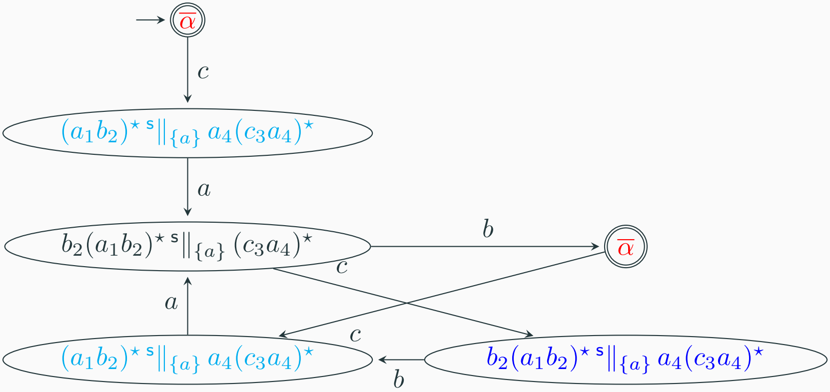
$$\mathsf{Follow}(\overline{\alpha}_1\ {}^{\mathsf{s}}\|_{\Gamma}\ \overline{\alpha}_2, (p_1, p_2)) = \bigcup_{\substack{\sigma \in \Sigma \\ [\mathsf{b}_1, \mathsf{b}_2] \in s_\sigma}} \{(\sigma, (q_1, q_2)) \mid (\sigma, q_i) \in \mathsf{Follow}(\alpha_i, p_i) \text{ if } \mathsf{b}_i; q_i = p_i \text{ otherwise}\}$$

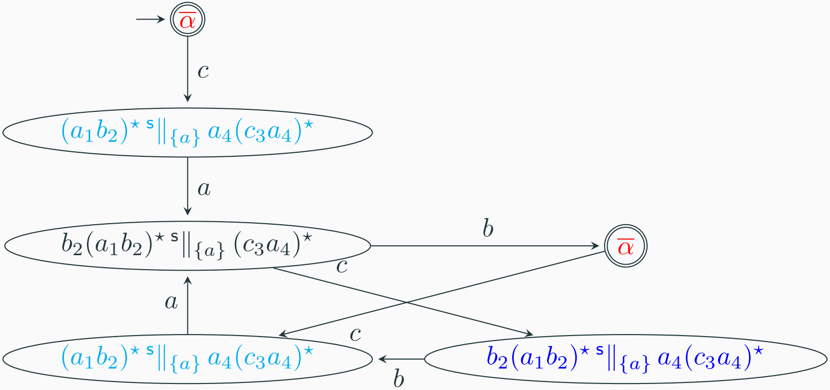
where $s_\sigma = \{\langle \top, \top \rangle\}$ if $\sigma \in \Gamma$; $s_\sigma = \{\langle \top, \bot \rangle, \langle \bot, \top \rangle\}$ otherwise.

$$\mathbb{B} = \{\top, \bot\},\ s_a = \{\langle \top, \top \rangle\} \text{ and } s_b = s_c = \{\langle \bot, \top \rangle, \langle \top, \bot \rangle\}$$



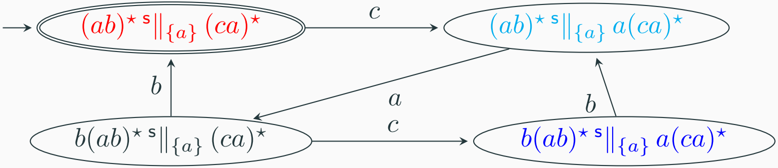
$$\overline{\alpha} = (a_1 b_2)^{\star} \;{}^{\mathsf{s}}\|_{\{a\}}\; (c_3 a_4)^{\star}$$





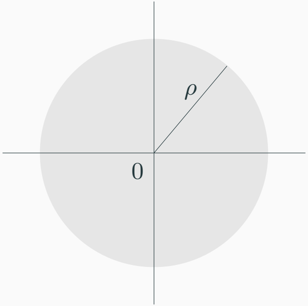
$$p \equiv_c q \text{ if } \overline{\mathsf{c}(\overline{\alpha}, p)} = \overline{\mathsf{c}(\overline{\alpha}, q)}$$

$\mathcal{A}_{\mathcal{G}\ell}(\alpha)/_{\equiv_c} \simeq \mathcal{A}_{\mathrm{PD}}(\alpha)$



- Uniform distribution of regular expressions of size n and $|\Sigma| = k$
- Framework of analytic combinatorics
- Relates combinatorial objects to algebraic properties of complex analytic generating functions
- the behaviour of these functions around their dominant singularities gives access to the asymptotic form of their (power) series coefficients.

$$C(z) = \sum_n c_n z^n$$



$$c_n \sim \theta_n \rho^{-n}$$

where θ_n is a sub-exponential

- $k = |\Sigma|$
- $R_k(z)$, generating function for regular expressions α with $n = |\alpha|$
- $P_k(z)$, generating function for an upper bound of the size of Q_{PD}

$$avP = \frac{[z^n]P_k(z)}{[z^n]R_k(z)}$$

	Measure	Average C.	Worst C.	Ref. Avg
RE($\sqcup$)	$\lvert\{\partial_\Sigma(\alpha)\}\rvert$	3	$O(n)$	[BMMR23a]
	$\lvert Q_{\mathrm{PD}}\rvert$	$2\sqrt{3}(\frac{4}{3})^{\frac{n}{2}}$	$O(2^n)$	[BMMR18]
RE($\cap$)	$\lvert\{\partial_\Sigma(\alpha)\}\rvert$	15	$O(n^2)$	[BMMR23a]
	$\lvert Q_{\mathrm{PD}}\rvert$	3	$O(2^n)$	[BMMR23a]
RE($\parallel$)	$\lvert\{\partial_\Sigma(\alpha)\}\rvert$	9	$O(n^2)$	[BMMR23a]
	$\lvert Q_{\mathrm{PD}}\rvert$	$(1.12859 + o(1))^n$	$O(2^n)$	[BMMR23a]
RE($\lvert\lvert\lvert$)	$\lvert\{\partial_\Sigma(\alpha)\}\rvert$	$\begin{cases} 8 & \lvert\Sigma\rvert \geq 4 \\ 7.5 \times (1.00528)^n & \lvert\Sigma\rvert = 2 \\ 18.7 \times (1.00062)^n & \lvert\Sigma\rvert = 3 \end{cases}$	$O(n^3)$	[BMMR23a]
	$\lvert Q_{\mathrm{PD}}\rvert$	$0.26 \times (1.35488 + o(1))^n$	$O(3^n)$	[BMMR25]

- $\mathbb{B} = \{\top, \bot\}$
- $\mathcal{F}_n$ set of Boolean functions $f : \mathbb{B}^n \longrightarrow \mathbb{B}$
- $\mathsf{sat}(f) = \{\langle \mathsf{b}_1, \ldots, \mathsf{b}_n \rangle \in \mathbb{B}^n \mid f(\mathsf{b}_1, \ldots, \mathsf{b}_n) = \top\} \setminus \{\langle \bot, \ldots, \bot \rangle\}$.
- $\psi : \Sigma \longrightarrow \mathcal{F}_n$

$$
\begin{aligned}
\|_\psi\,(\varepsilon, \ldots, \varepsilon) &= \{\varepsilon\} \\
\|_\psi\,(u_1, \ldots, u_n) &= \{\sigma w \mid (\exists \mathsf{b} \in \mathsf{sat}(\psi(\sigma))) \\
&\qquad\qquad [u_i = \sigma v_i \text{ if } \mathsf{b}_i; u_i = v_i \text{ otherwise}] \\
&\qquad\qquad \wedge w \in \|_\psi\,(v_1, \ldots, v_n)\}.
\end{aligned}
$$

The operator $\|_\psi$ is extended to languages as usual by

$$
\|_\psi\,(L_1, \ldots, L_n) = \bigcup_{u \in L_1 \times \cdots \times L_n} \|_\psi\,(u_1, \ldots, u_n),
$$

where $u = (u_1, \ldots, u_n)$.

- $\|_{\wedge}(L_1, L_2) = L_1 \cap L_2$
- $\|_{\oplus}(L_1, L_2) = L_1 \sqcup L_2$
- $\|_{\vee}(L_1, L_2) = L_1 \,{}^{\mathsf{a}}\|_{\Sigma}\, L_2$
- $\|_{\psi_s}(L_1, L_2) = L_1 \,{}^{\mathsf{s}}\|_{\Gamma}\, L_2$ where

$$\psi_s(\sigma) = \begin{cases} \wedge, & \text{if } \sigma \in \Gamma, \\ \oplus, & \text{otherwise,} \end{cases}$$

- $\|_{\psi_a}(L_1, L_2) = L_1 \,{}^{\mathsf{a}}\|_{\Gamma}\, L_2$

$$\psi_a(\sigma) = \begin{cases} \vee, & \text{if } \sigma \in \Gamma, \\ \oplus, & \text{otherwise.} \end{cases}$$

- $\|_{\circ}(L_1, L_2, L_3) = L_1 \,|\!\|_{L_2}\, L_3$ where $\circ(\mathsf{b}_1, \mathsf{b}_2, \mathsf{b}_3) = (\mathsf{b}_2 \leftrightarrow (\mathsf{b}_1 \wedge \mathsf{b}_3))$

$$\mathcal{A}_i = (Q_i, \Sigma, \delta_i, \iota_i, F_i)$$

$(i = 1, \ldots, n)$

$$\|(\mathcal{A}_1, \ldots, \mathcal{A}_n) = (Q, \Sigma, \delta_\psi, (\iota_1, \ldots, \iota_n), F)$$

where

- $Q = Q_1 \times \cdots \times Q_n$
- $F = F_1 \times \cdots \times F_n$,
- $s = (s_1, \ldots, s_n), t = (t_1, \ldots, t_n)$, $\mathsf{b} = (\mathsf{b}_1, \ldots, \mathsf{b}_n)$, and

$$\delta_\psi(s, \sigma) = \bigcup_{\mathsf{b} \in \mathsf{sat}(\psi(\sigma))} \{t \in Q \mid (\forall i)[t_i \in \delta_i(s_i, \sigma) \text{ if } \mathsf{b}_i; t_i = s_i \text{ otherwise}]\}$$

If L_i are regular

- $\|_\psi (L_1, \ldots, L_n)$ are regular
- If L_i are defined by regular expressions α_i one can define
  - $\mathcal{A}_{\mathrm{PD}}(\|_\psi (\alpha_1, \ldots, \alpha_n))$
  - $\mathcal{A}_{\mathscr{G}\ell}(\|_\psi (\alpha_1, \ldots, \alpha_n))$
  - $\ldots$

  as Boolean Shuffle (Product) Automata.

- allow to express different compositional behaviours of concurrent systems
- unique framework for generalised synchronised shuffles and several automata constructions

Thank you for your attention

V. M. Antimirov.
Partial derivatives of regular expressions and finite automaton constructions.
Theoret. Comput. Sci., 155(2):291–319, 1996.

Jean Berstel, Luc Boasson, Olivier Carton, Jean-Eric Pin, and Antonio Restivo.
The expressive power of the shuffle product.
Inf. Comput., 208(11):1258–1272, 2010.

Franziska Biegler, Mark Daley, and Ian McQuillan.
On the shuffle automaton size for words.
J. Autom. Lang. Comb., 15(1/2):53–70, 2010.

Janusz A. Brzozowski, Galina Jirásková, Bo Liu, Aayush Rajasekaran, and Marek Szykula.
On the state complexity of the shuffle of regular languages.

In Cezar Câmpeanu, Florin Manea, and Jeffrey O. Shallit, editors, *18th DCFS, 2016*, volume 9777 of *LNCS*, pages 73–86. Springer, 2016.

Franziska Biegler and Ian McQuillan.
On comparing deterministic finite automata and the shuffle of words.
In Markus Holzer and Martin Kutrib, editors, *Implementation and Application of Automata - 19th International Conference, CIAA 2014, Giessen, Germany, July 30 - August 2, 2014. Proceedings*, volume 8587 of *LNCS*, pages 98–109. Springer, 2014.

Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis.
Position automaton construction for regular expressions with intersection.
In Srecko Brlek and Christophe Reutenauer, editors, *20th DLT 2016*, volume 9840 of *LNCS*, pages 51–63. Springer, 2016.

Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis.
Automata for regular expressions with shuffle.
Inf. Comput., 259(2):162–173, 2018.

Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis.
Location based automata for expressions with shuffle.
In Alberto Leporati, Carlos Martín-Vide, Dana Shapira, and Claudio Zandron, editors, *15th LATA 2021*, volume 12638 of *LNCS*, pages 43–54. Springer, 2021.

Sabine Broda, Eva Maia, Nelma Moreira, and Rogério Reis.
The prefix automaton.
J. Autom. Lang. Comb., 26(1-2):17–53, 2021.

Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis.
Average complexity of partial derivatives for synchronised shuffle expressions.

In Benedek Nagy, editor, *27th CIAA*, volume 14151 of *LNCS*, pages 103–115. Springer, 2023.

Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis.
Location automata for synchronised shuffle expressions.
J. Log. Algebr. Program., 133, 2023.

Sabine Broda, António Machiavelo, Nelma Moreira, and Rogério Reis.
Average complexity of partial derivatives for synchronised shuffle expressions.
In Luca Prigioniero and Robert Merkas, editors, *26th DCFS*, LNCS. Springer, 2025.

Jan A. Bergstra, Alban Ponse, and Scott A. Smolka, editors.
Handbook of Process Algebra.
North-Holland / Elsevier, 2001.

Pascal Caron, Jean-Marc Champarnaud, and Ludovic Mignot.

Multi-tilde-bar expressions and their automata.
Acta Informatica, 49(6):413–436, 2012.

K.T. Chen, R.H. Fox, and R.C. Lyndon.
Free differential calculus, iv. the quotient groups of the lower central series.
Annals of Mathematics, 68(1):81–95, 1958.

Pascal Caron, Jean-Gabriel Luque, and Bruno Patrou.
A combinatorial approach for the state complexity of the shuffle product.
J. Autom. Lang. Comb., 25(4):291–320, 2020.

Cezar Câmpeanu, Kai Salomaa, and Sheng Yu.
Tight lower bound for the state complexity of shuffle of regular languages.
J. Autom. Lang. Comb., 7(3):303–310, 2002.

J. M. Champarnaud and D. Ziadi.
Canonical derivatives, partial derivatives and finite automaton constructions.
Theoret. Comput. Sci., 289:137–163, 2002.

Robert de Simone.
Langages infinitaires et produit de mixage.
Theor. Comput. Sci., 31:83–100, 1984.

Samuel Eilenberg and Saunders Mac Lane.
On the groups h(π, n).
Annals of Mathematics, 58(1):55–106, 1953.

Wouter Gelade.
Succinctness of regular expressions with interleaving, intersection and counting.
Theor. Comput. Sci., 411(31-33):2987–2998, 2010.

Hermann Gruber and Markus Holzer.
Tight bounds on the descriptional complexity of regular expressions.
In Volker Diekert and Dirk Nowotka, editors, *13th Developments in Language Theory (DLT 2009)*, volume 5583 of *LNCS*, pages 276–287. Springer, 2009.

V. M. Glushkov.
The abstract theory of automata.
Russ. Math. Surv., 16:1–53, 1961.

Seymour Ginsburg and Edwin H. Spanier.
Mappings of languages by two-tape devices.
J. ACM, 12(3):423–434, 1965.

L. Ilie and S. Yu.
Follow automata.
Inf. Comput., 186(1):140–162, 2003.

Michel Latteux and Yves Roos.
Synchronized shuffle and regular languages.
In Juhani Karhumäki, Hermann A. Maurer, Gheorghe Paun, and Grzegorz Rozenberg, editors, *Jewels are Forever, Contributions on Theoretical Computer Science in Honor of Arto Salomaa*, pages 35–44. Springer, 1999.

Florin Manea, Victor Mitrana, and Daniel-Claudian Voinescu.
Synchronized shuffle on backbones.
Fundam. Inform., 73(1-2):191–202, 2006.

Alain J. Mayer and Larry J. Stockmeyer.
Word problems-this time with interleaving.
Inf. Comput., 115(2):293–311, 1994.

William F. Ogden, William E. Riddle, and William C. Rounds.
Complexity of expressions allowing concurrency.

In Alfred V. Aho, Stephen N. Zilles, and Thomas G. Szymanski, editors, *Conference Record of the Fifth Annual ACM Symposium on Principles of Programming Languages,*, pages 185–194. Tucson, Arizona, USA, ACM Press, 1978.

Holger Petersen.
The membership problem for regular expressions with intersection is complete in LOGCFL.
In Helmut Alt and Afonso Ferreira, editors, *STACS 2002, 19th Annual Symposium on Theoretical Aspects of Computer Science, Antibes - Juan les Pins, France, March 14-16, 2002, Proceedings*, volume 2285 of *LNCS*, pages 513–522. Springer, 2002.

Grzegorz Rozenberg and Arto Salomaa, editors.
Handbook of Formal Languages.
Springer, 1997.

Jacques Sakarovitch and Imre Simon.
Subwords.
page 105.

Martin Sulzmann and Peter Thiemann.
Derivatives and partial derivatives for regular shuffle expressions.
J. Comput. Syst. Sci., 104:323–341, 2019.

Maurice H. ter Beek, Clarence A. Ellis, Jetty Kleijn, and Grzegorz Rozenberg.
Synchronizations in team automata for groupware systems.
Comput. Support. Cooperative Work., 12(1):21–69, 2003.

Maurice H. ter Beek and Jetty Kleijn.
Team automata satisfying compositionality.
In Keijiro Araki, Stefania Gnesi, and Dino Mandrioli, editors, *FME 2003: Formal Methods, International Symposium of Formal Methods Europe, Pisa,*

Italy, September 8-14, 2003, Proceedings, volume 2805 of *LNCS*, pages 381–400. Springer, 2003.

Maurice H. ter Beek, Carlos Martín-Vide, and Victor Mitrana.
Synchronized shuffles.
Theoret. Comput. Sci., 341:263–275, 2005.

Hiroaki Yamamoto.
A new finite automaton construction for regular expressions.
In Suna Bensch, Rudolf Freund, and Friedrich Otto, editors, *6th NCMA*, volume 304 of *books@ocg.at*, pages 249–264. Österreichische Computer Gesellschaft, 2014.