From text indexing to regular language indexing

One FLAT World Seminar

November 26, 2025

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Università
Ca’Foscari
Venezia

Overview

1. Sorting and indexing

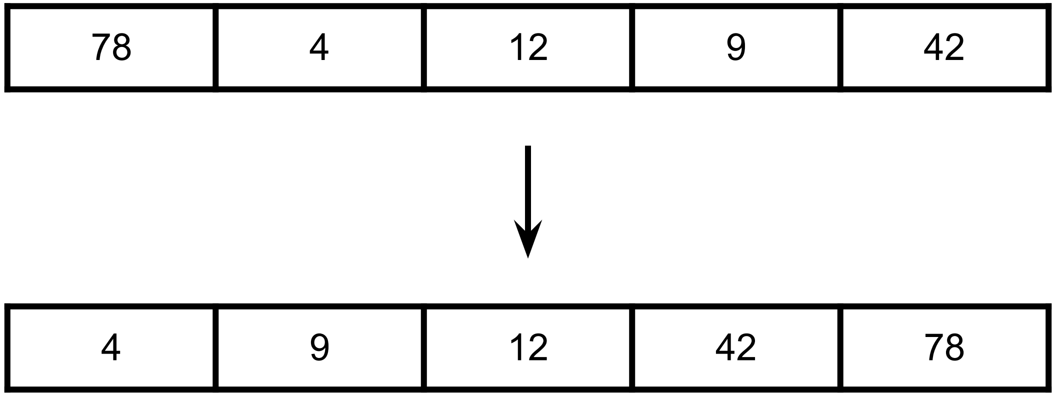
2. Wheeler automata / languages

3. p-sortable automata / languages

Sorting and indexing

Sorting

Sorting is the algorithmic process of ordering the elements of a given set according to a specific order.



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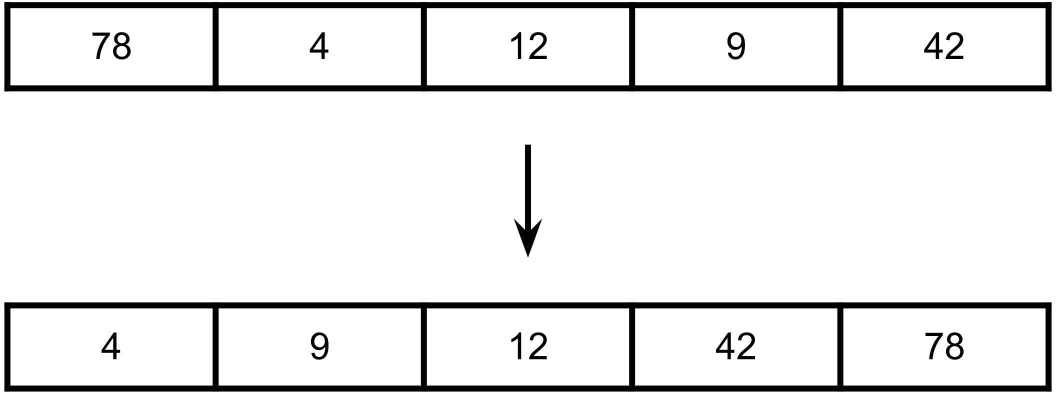
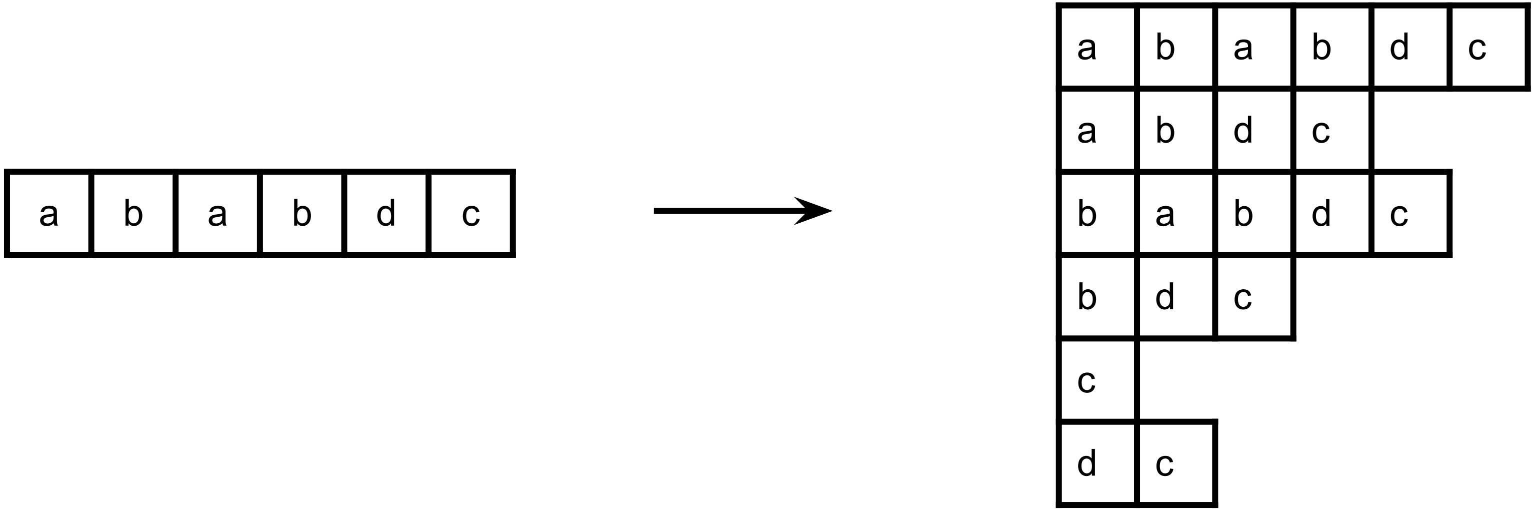


Benefits: the sorted list is

- Searchable (binary search)
- More compressible (store just the differences between consecutive integers)

Sorting

Not just integers. Other example: suffixes of a string



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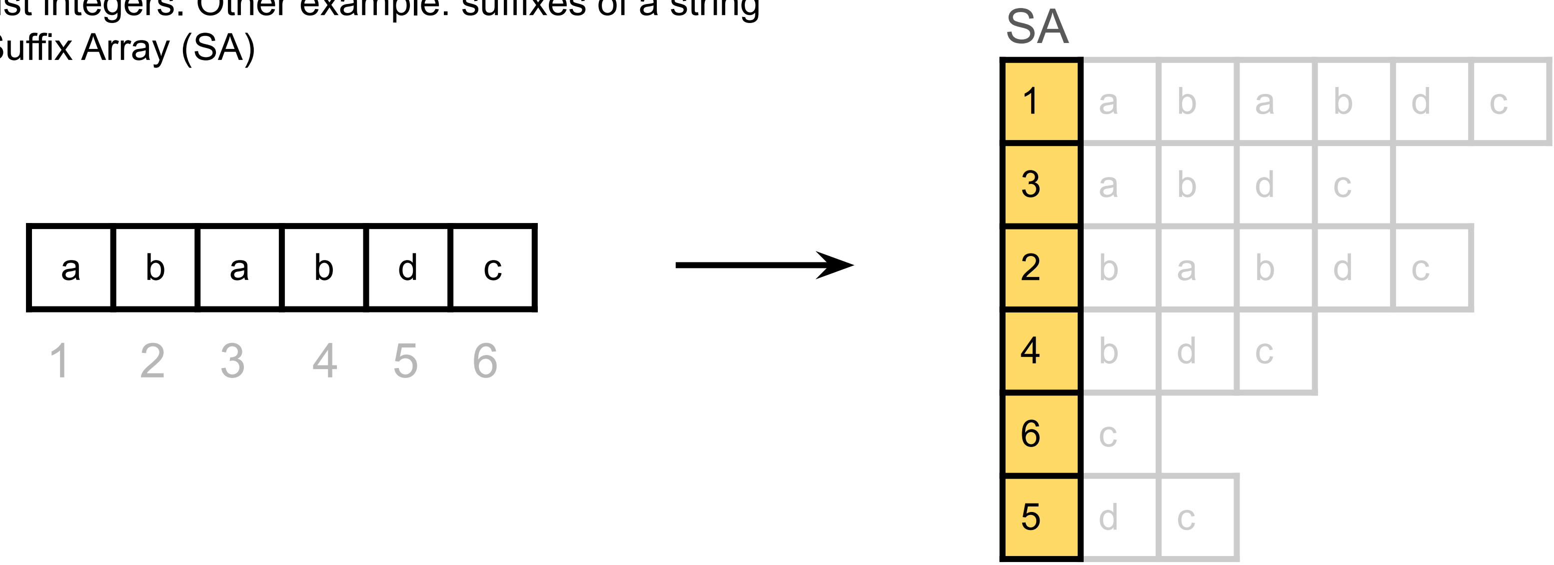

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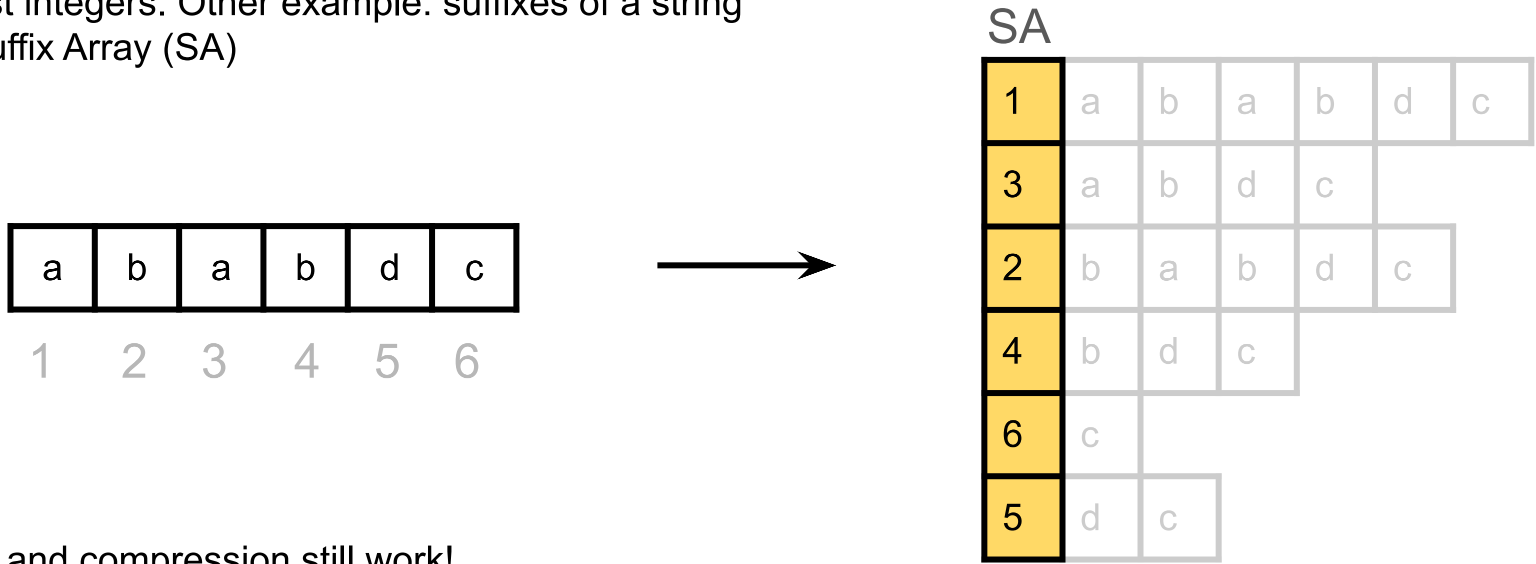

Indexing and compression still work!

- **Indexing**: suffixies prefixed by a word (e.g. “ab”) form a range. Can be found by binary search on SA.

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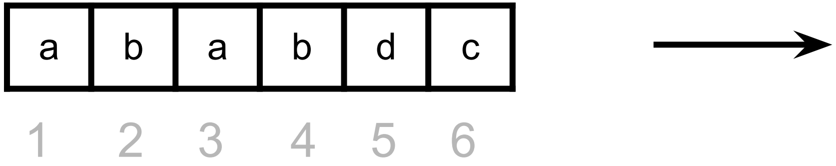

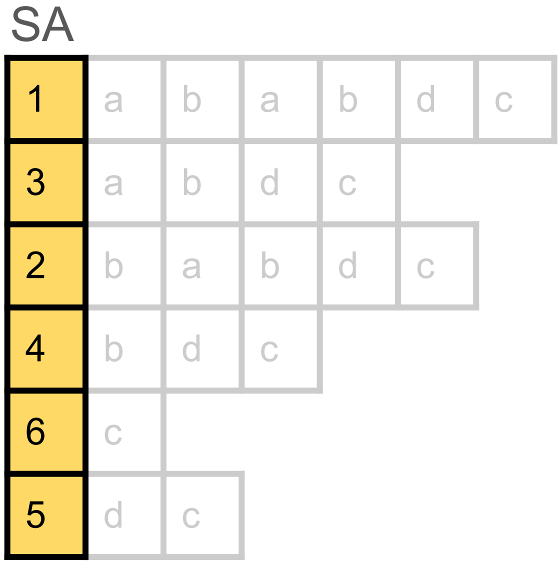

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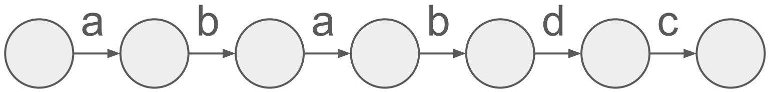
- **Indexing**: suffixies prefixed by a word (e.g. “ab”) form a range. Can be found by binary search on SA.
- SA can be **compressed** while still supporting fast pattern matching.

Roberto Grossi, Jeffrey Scott Vitter, 2000. Compressed Suffix Arrays and Suffix Trees with Applications to Text Indexing and String Matching. STOC.
Paolo Ferragina, Giovanni Manzini, 2000. Opportunistic Data Structures with Applications. FOCS.

Text indexing

Observations:

- A string is a very simple kind of NFA



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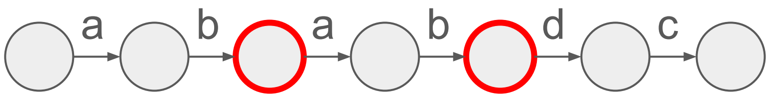

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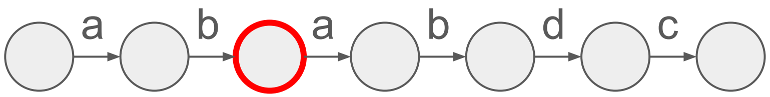

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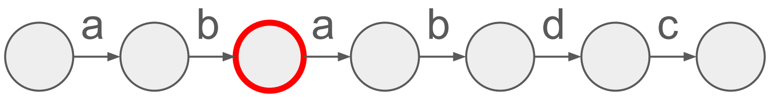


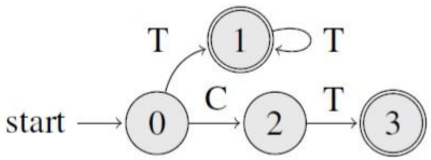
Can we generalize suffix sorting to regular languages?
Ideally: queries in O(|P|) time, where P is the query string.

Problem definition

Problem (regular language indexing)

Given a NFA, build a (small) data structure supporting efficiently the following queries:

- Count: given string P, return the number of states reached by a path labeled P
- Locate: given string P, return the states reached by a path labeled P
- Membership: given string P, is a final state reached by a path labeled P starting in the source?



Example:

- count("T") = 2, locate("T") = {1,3}
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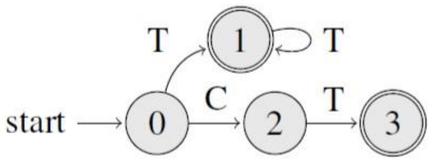

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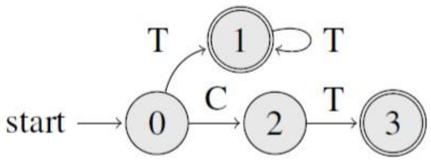

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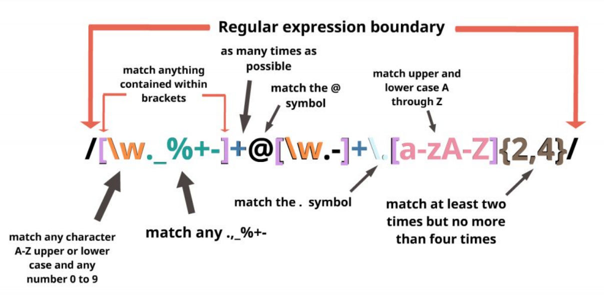

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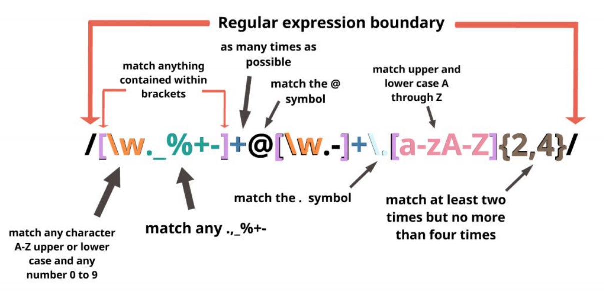
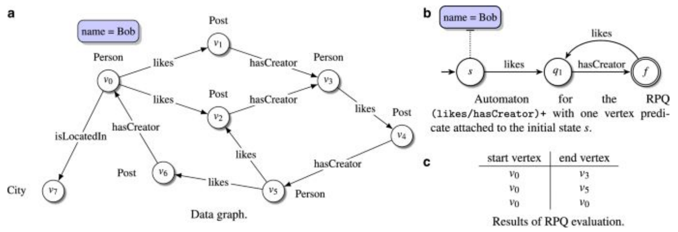


- Graph databases (e.g. regular path queries) (matching a regex on a graph)



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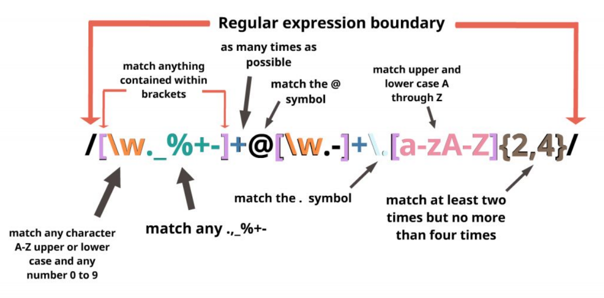

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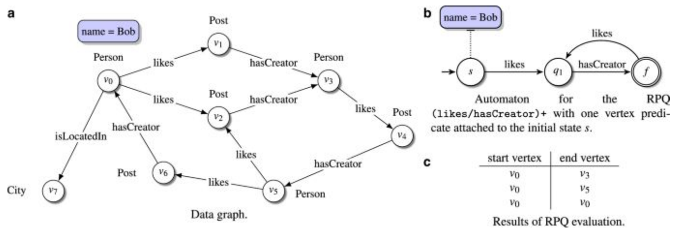
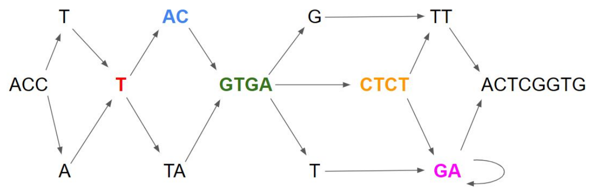


- Bioinformatics (pattern matching on pan-genome graphs). (matching a text on graph)



Lower bounds

Unfortunately, unless the Strong Exponential Time Hypothesis (SETH) fails:

- Matching a regular expression E on text T requires $\Omega(|E| \cdot |T|)$ time [Backurs, Indyk FOCS ‘16]

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- Regular path queries (emptiness) require quadratic time: size of graph times size of NFA for the regex [Casel, Schmid, LMCS’23]
- Solving pattern matching queries on labeled graphs requires $\Omega(m \cdot n)$ time, where m is the graph’s size and n is the query length [Equi et al. TALG’23]

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Massimo Equi, Veli Mäkinen, Alexandru Tomescu, Roberto Grossi (2023). On the complexity of string matching for graphs. *ACM Transactions on Algorithms*, *19*(3), 1-25.

Not hard on all NFA!

Although these problems are hard in general, for some NFA they are easy: paths, trees, de Bruijn graphs, …

Next slides: what is the most general class of “indexable/sortable” NFA?

- The class of Wheeler NFA / languages
- Generalization to arbitrary NFA

Wheeler automata
and
Wheeler languages

Myhill-Nerode relation

- We take our first steps from a central object in finite automata theory: the MN equivalence relation
- Intuition: we will turn the MN relation into an order, and use it to index the NFA

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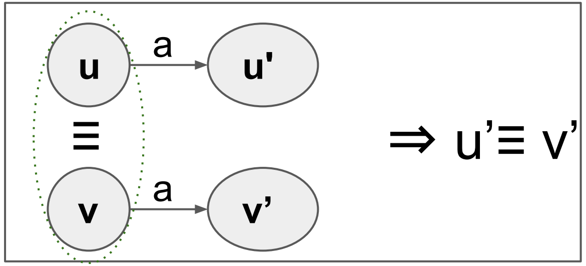

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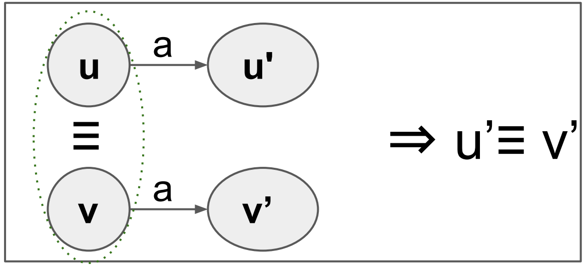

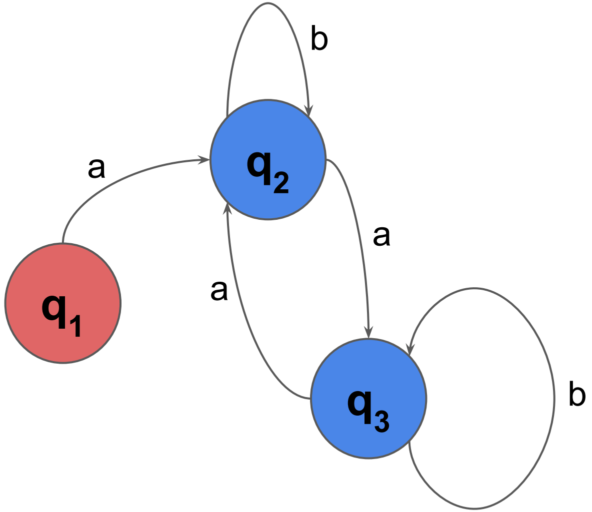

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Switching to an order

… but to index, we need an **order**. What if we turn ≡ (equivalence relation) into a **total order** ≤?

Equivalence relation ≡

reflexive:	$x \equiv x$
symmetric:	$x \equiv y \Leftrightarrow y \equiv x$
transitive:	$x \equiv y \land y \equiv z \Rightarrow x \equiv z$

Total order ≤

reflexive:	$x \leq x$
antisymmetric:	$x \leq y \land y \leq x \Rightarrow x = y$
transitive:	$x \leq y \land y \leq z \Rightarrow x \leq z$
strongly connected:	$x \leq y \lor y \leq x$

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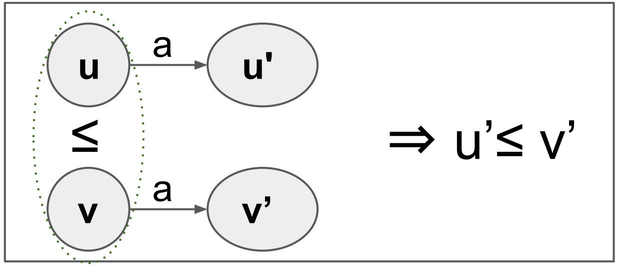
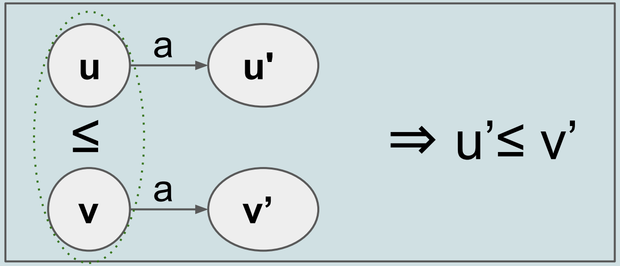


Ordered Automata

We obtain:

Def: ordered automaton (OA)
An OA is a NFA for which there exists a total order ≤ satisfying:



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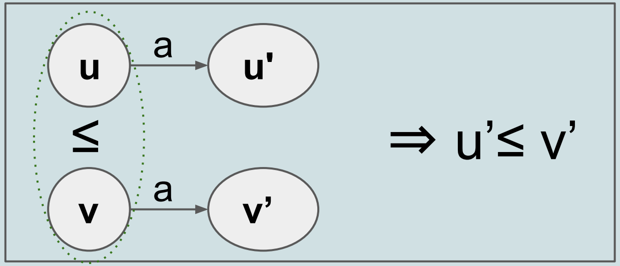
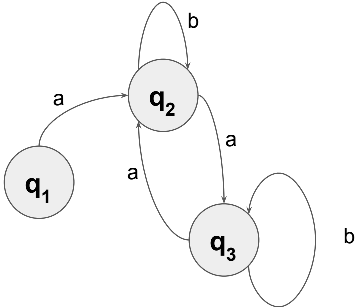


Such a total order does not always exist!



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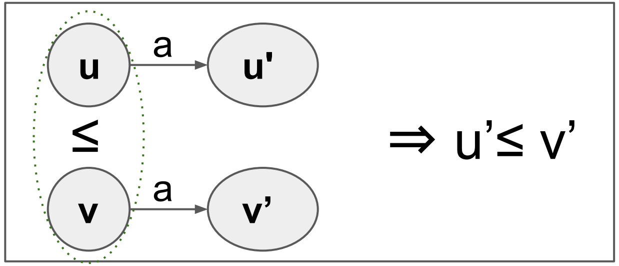

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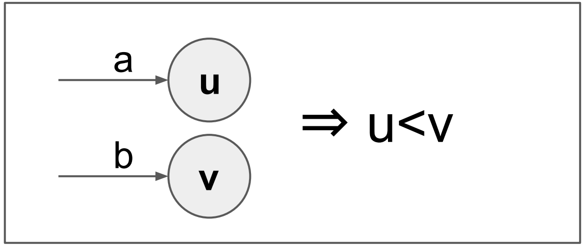

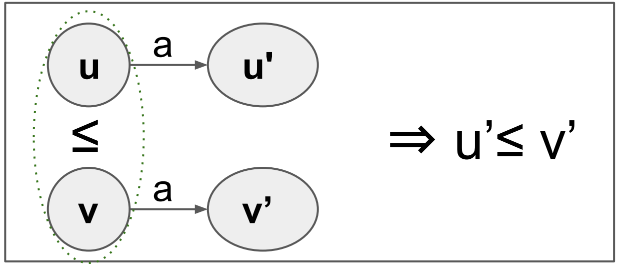

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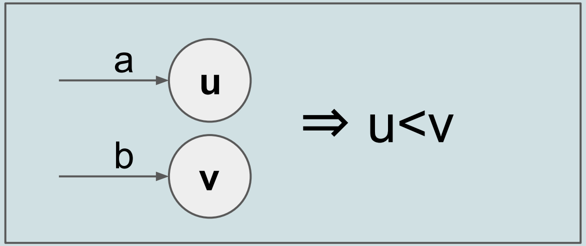

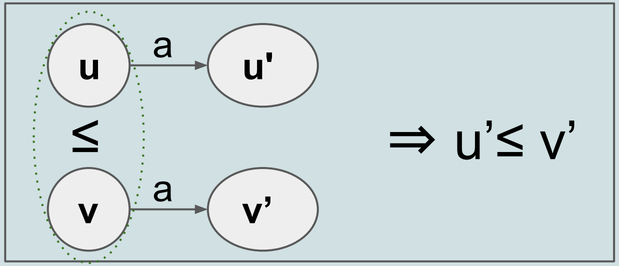

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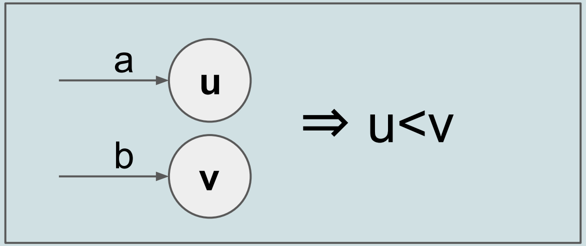

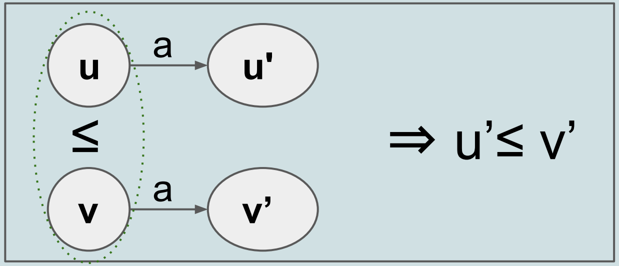

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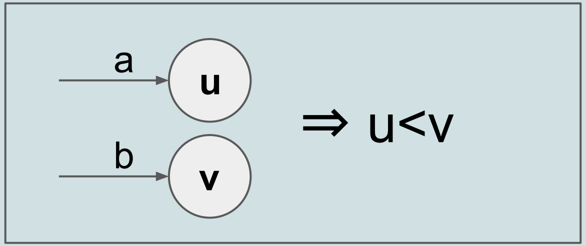

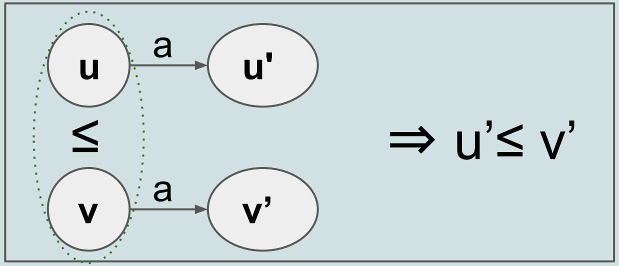

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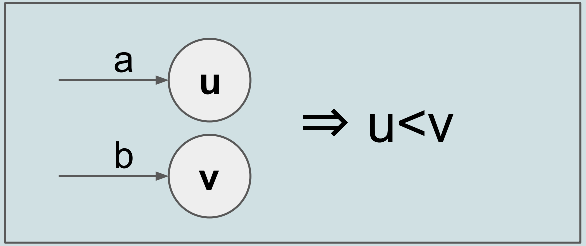

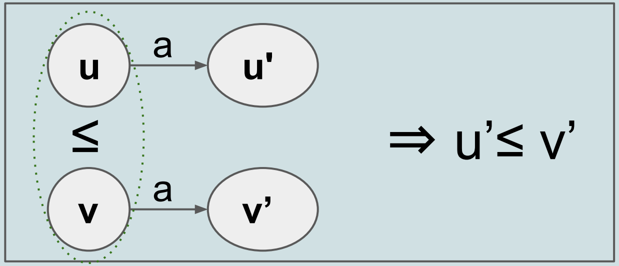

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*** Note: WNFA ⊂ OA ⇒ *Wheeler languages* are star-free

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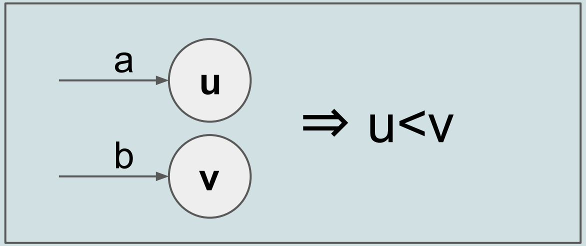

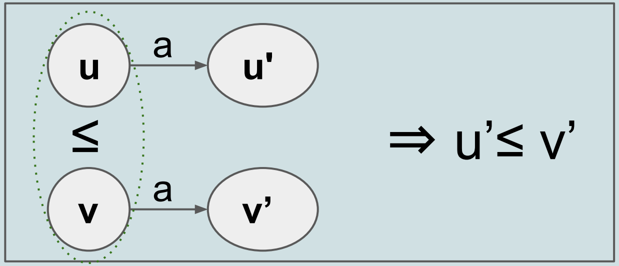

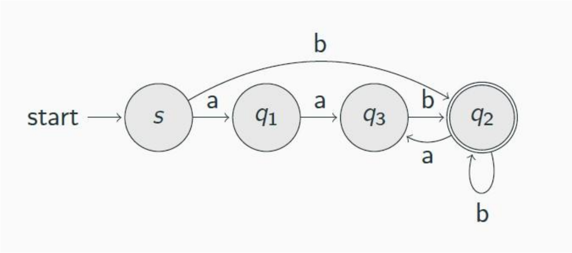

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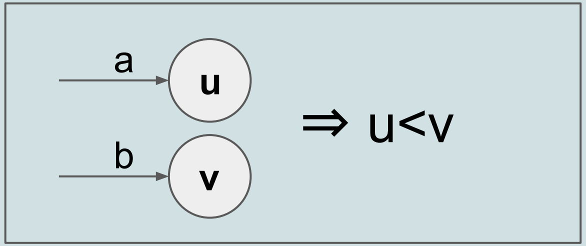
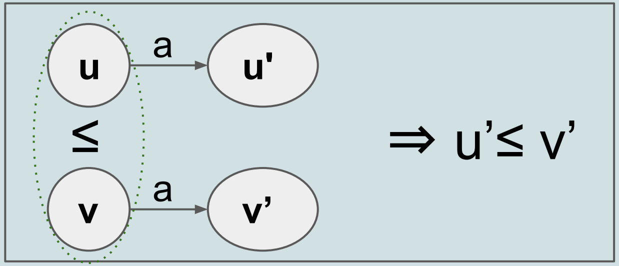




Axiom (1) makes things much more interesting w.r.t. Ordered Automata! Next slides:

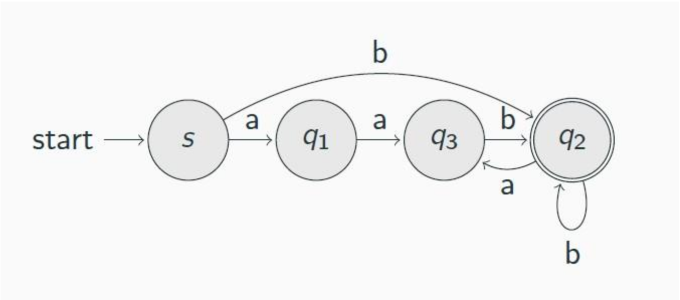
- Efficient encoding
- Linear-time queries
- Wheeler languages

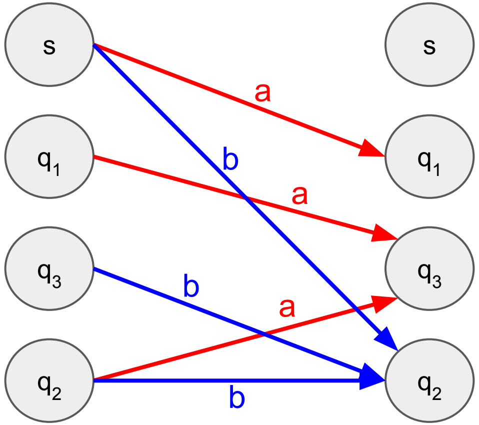
Gagie, Manzini, Sirén, 2017. "Wheeler graphs: A framework for BWT-based data structures." TCS

Bipartite representation

Useful visualization of the Wheeler order: **bipartite representation**.

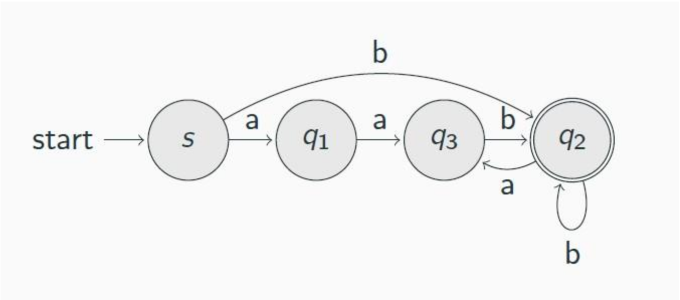
- Build a bipartite graph: two copies of the nodes, sorted by the candidate order.
- Same edges of the input NFA, but drawn left-to-right.
- The order is Wheeler ⇔ Same-letter edges must not cross.

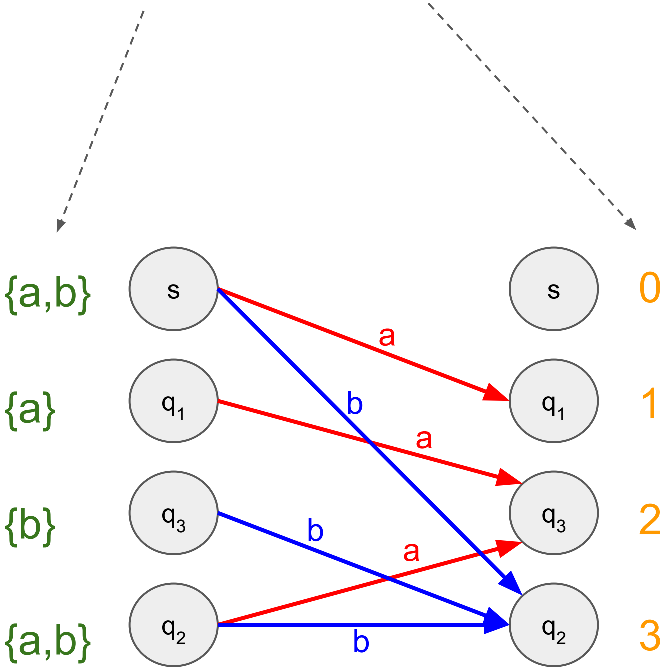




Efficient encoding

⇒ we can store the WNFA in O(1) bits per edge*! just store out-going labels and in-degrees



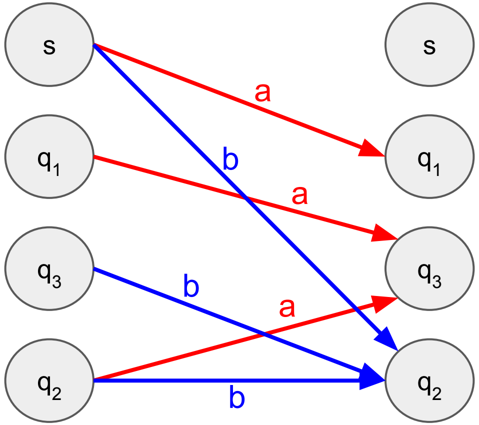


*assuming constant-size alphabet for simplicity

Pattern matching

Also:

same-letter edges must not cross
⇒
Nodes reached by a given string P form a *range* in the total order



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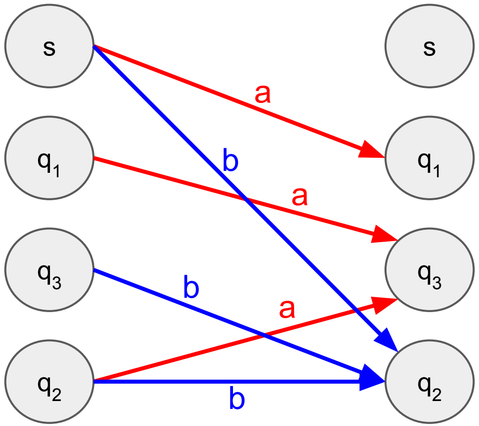


Consequences:

- WNFAs generalize known indexes on strings, trees, de Bruijn graphs…
- Indexed pattern matching/membership in optimal O(|P|) time

Pattern matching

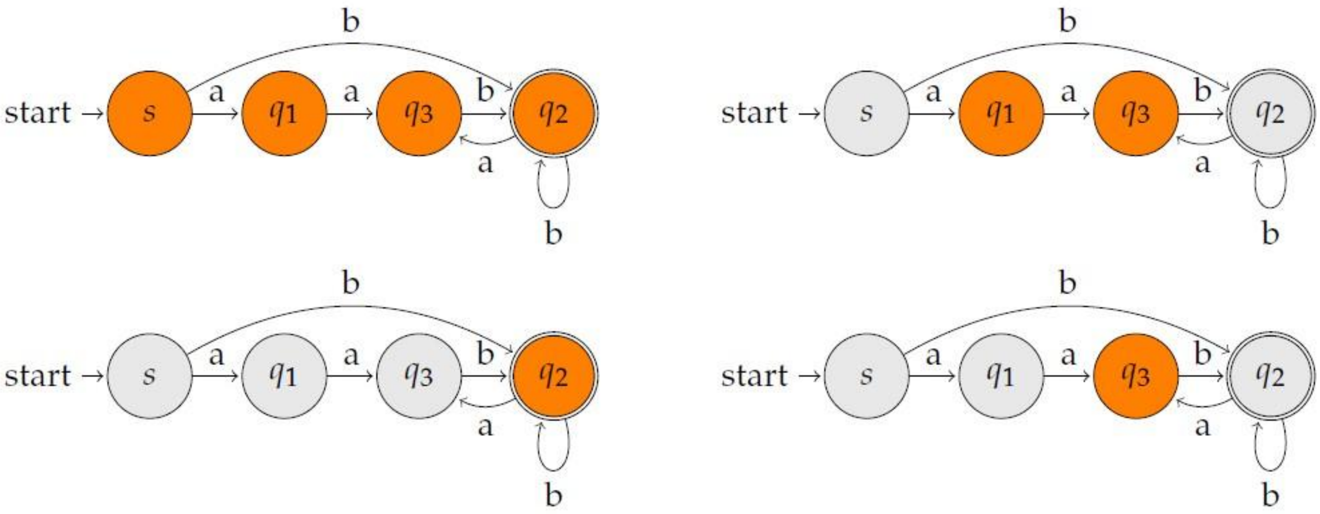
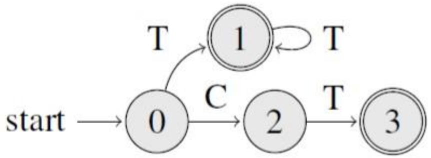


Figure 11. Searching nodes reached by a path labeled "aba" in a Wheeler graph. Top left: we begin with the nodes reached by the empty string (full range). Top right: range obtained from the previous one following edges labeled 'a'. Bottom left: range obtained from the previous one following edges labeled 'b'. Bottom right: range obtained from the previous one following edges labeled 'a'. This last range contains all nodes reached by a path labeled "aba"

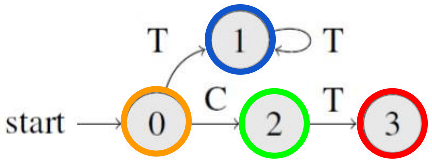
Wheeler languages



minimum DFA for L = TT* | CT

ϵ	<	C	<	T	<	CT	<	TT	<	TTT	<	TTTT	...
0		2		1		3		1		1		1	

Wheeler languages



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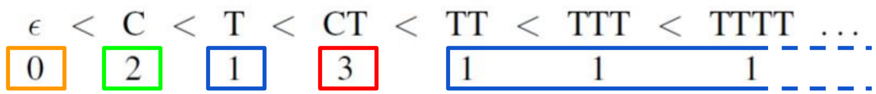
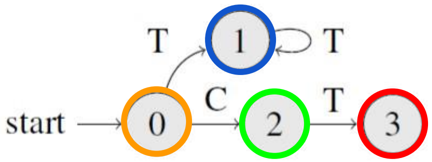


Finite number of Myhill-Nerode intervals in co-lex order

Wheeler languages



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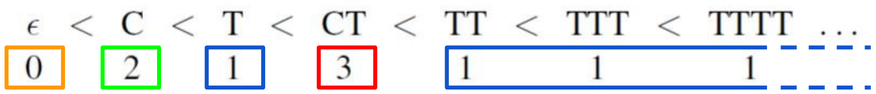

Finite number of Myhill-Nerode intervals in co-lex order ≡ **Wheeler language**

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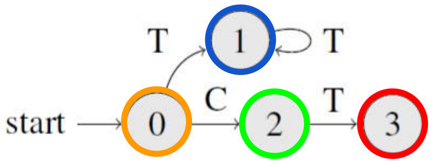
Wheeler languages



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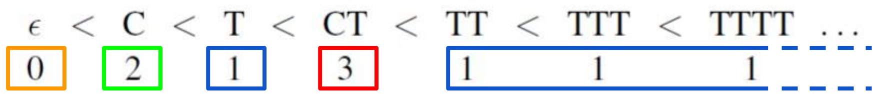

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Theorem [1] Myhill-Nerode theorem for W. languages. The following are equivalent:

1. *A regular language L is Wheeler*
2. *L is recognized by a Wheeler NFA*
3. *L is recognized by a Wheeler DFA*
4. *The Myhill-Nerode equivalence classes of L form a **finite number of intervals in co-lex order**.*

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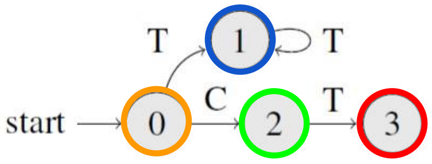

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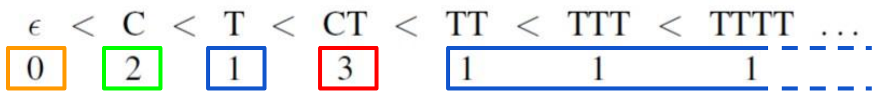

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In fact, given a WNFA we can always build an equivalent WDFA of at most twice the size!

[1] J. Alanko, G. D'Agostino, A. Policriti, and N. P. *Regular Languages meet Prefix Sorting*. SODA 2020.

Overview of algorithmic results on Wheeler languages

Overview of algorithmic results

m = size of input A
M = size of output

output / input	NFA	2-NFA	DFA	WNFA	WDFA
W. order of A if A is W					
min. equivalent WDFA if L(A) is W					
equivalent sorted WNFA if L(A) is W					
is L(A) Wheeler?					
Is A Wheeler?					

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Generalizing:
p-sortable automata and languages

Searching a partially-ordered set

Classic algorithmic result:

Given a set of objects and a partial order < on the set such that width(<) = p, then:

- Searching the set requires at least p log (n/p) operations
- There exists a data structure supporting search in time O(p log (n/p))

Intuition: decompose < into p totally-sorted chains (Dilworth's theorem), run binary search on each chain.

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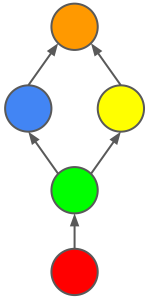
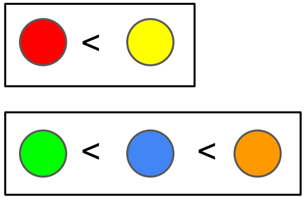
Hasse diagram of <

width(<) = 2





A chain decomposition into p=2 chains

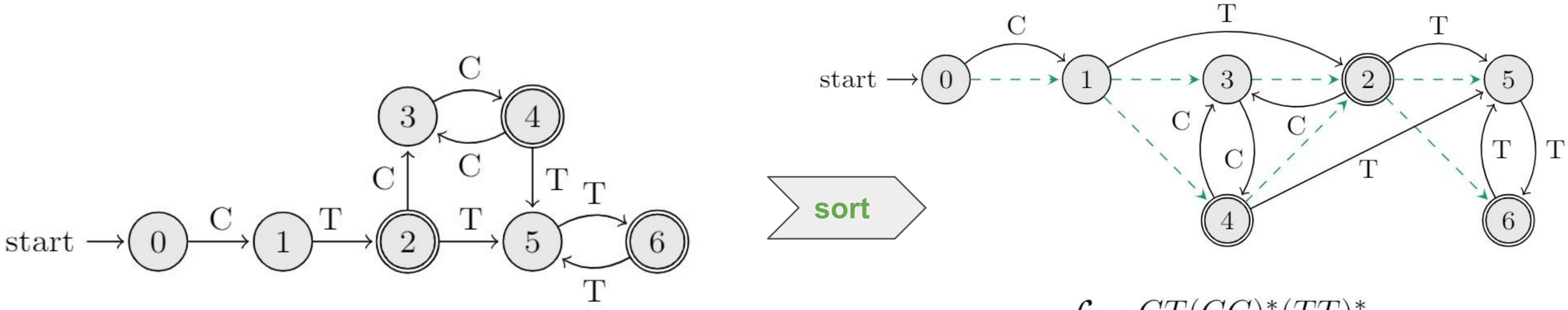
Generalization to arbitrary NFA: co-lex orders

For arbitrary NFAs: same idea of the Wheeler case, but do not require that $<$ is total.

Generalization to arbitrary NFA: co-lex orders

For arbitrary NFAs: same idea of the Wheeler case, but do not require that < is total.

Any NFA admits a ***partial co-lex order*** of its nodes.
We are interested in a **minimum-width** one (Wheeler NFA are the case width=1)



$$\mathcal{L} = CT(CC)^*(TT)^*$$

- - - → Hasse diagram

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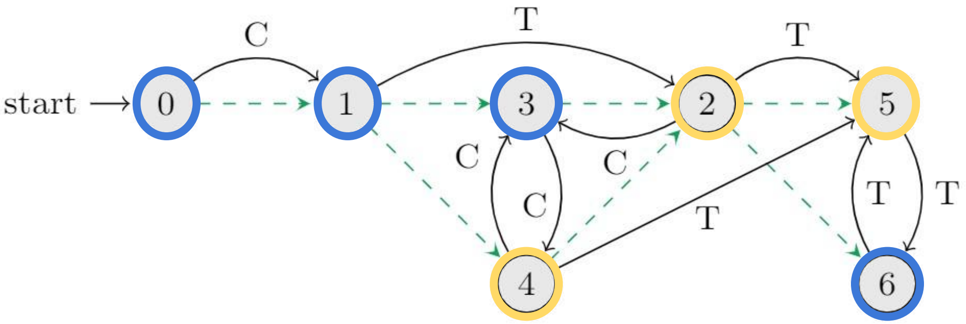
co-lex orders



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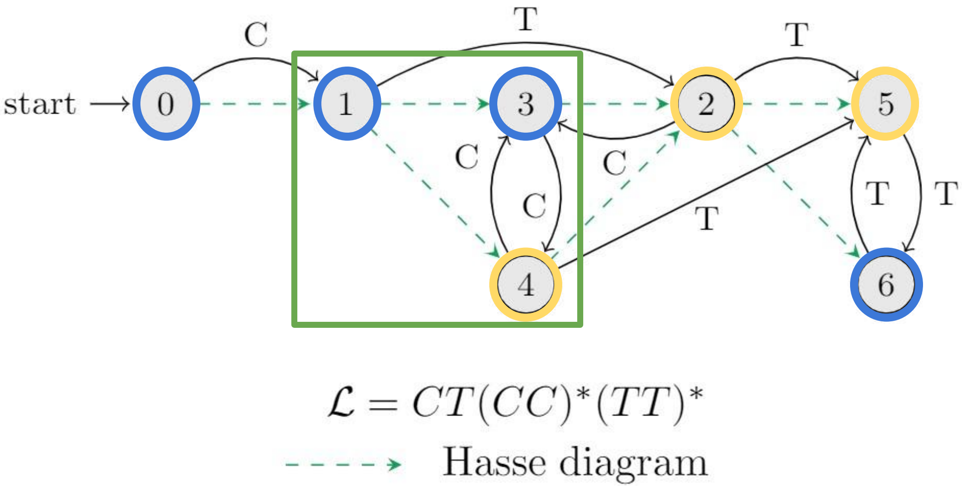
Hasse diagram

A possible chain partitioning (yellow, blue) of the partial order.

co-lex orders



A possible chain partitioning (yellow, blue) of the partial order.

Indexing ≡ states reached by any string (in the example, “C”) always form a *convex set in the partial order*.

co-lex orders

Let n = number of states.

Results. $p = \text{width}(<)$ is an important parameter for NFAs:

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- NFA of size n and width p ⇒ powerset construction ⇒ DFA of size ≤ $(\mathbf{n-p+1)\cdot 2^p}$ and width ≤ 2^p *

*consequence: NFA equivalence / universality (PSPACE-complete) are FPT w.r.t. p

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- Fast index construction with state-of-the-art algorithms.

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Team & Funding

Funded by ERC StG “REGINDEX: Compressed indexes for regular languages with applications to computational pan-genomics” grant nr 101039208.

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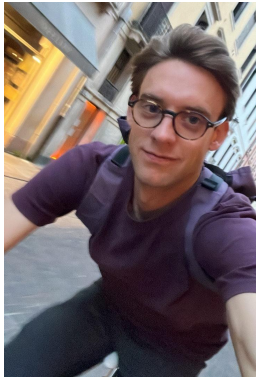







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