Adversarial Synchronization

Anton Lipin and Mikhail Volkov

February 25, 2026

Definitions and terminology

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To lighten notation we use qw for $\delta(q, w)$ and $\langle Q, \Sigma \rangle$ for $\langle Q, \Sigma, \delta \rangle$.

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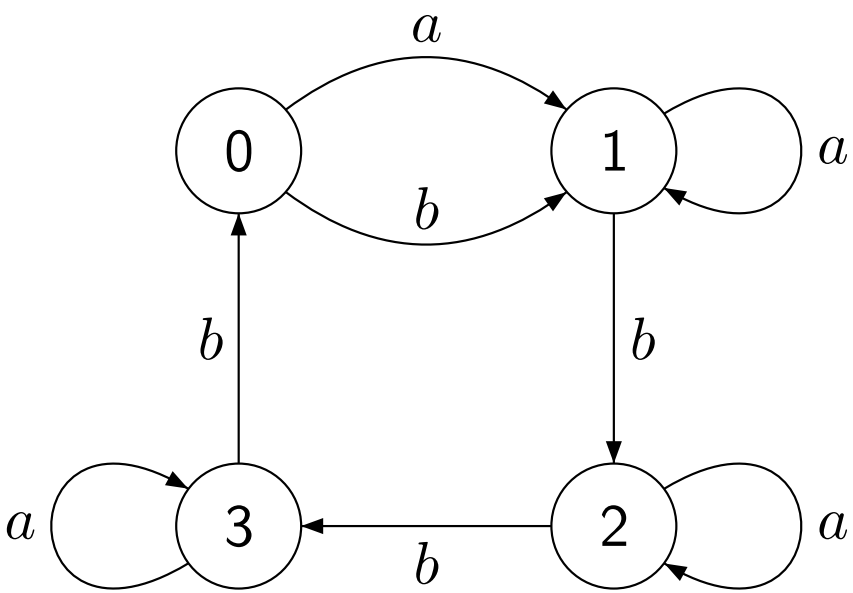

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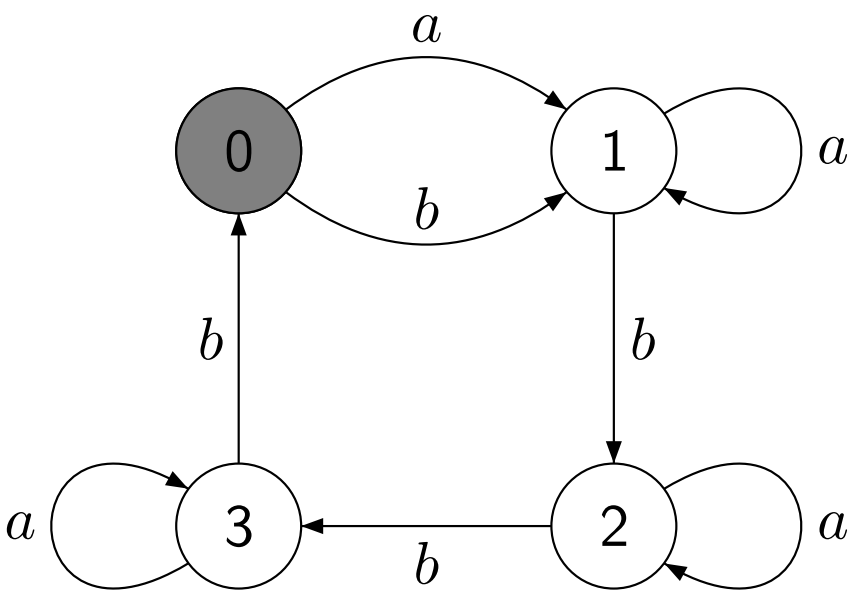

The fact that the automaton is in certain state can be visualized by putting a token on this state; say, now the automaton is in state 0.

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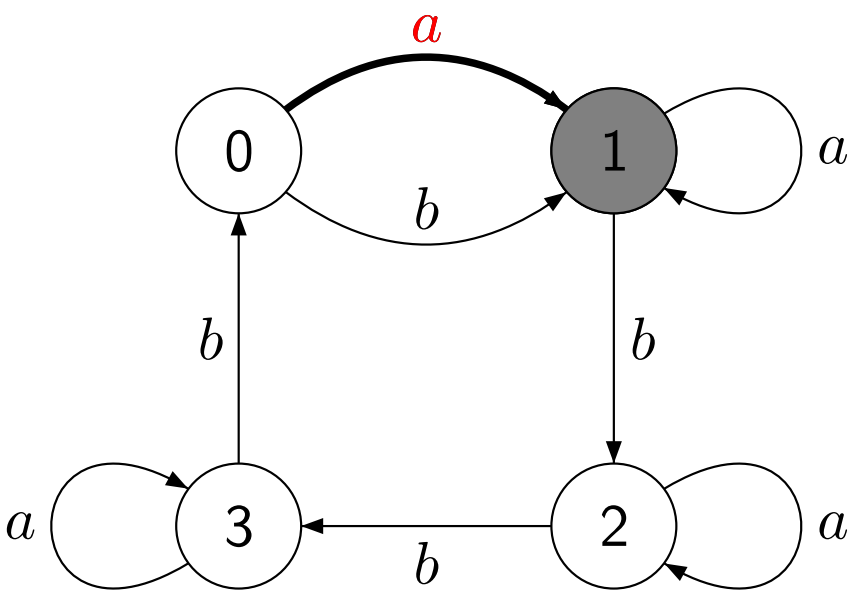

The action of a makes the token slide to state 1.

Synchronizing automata

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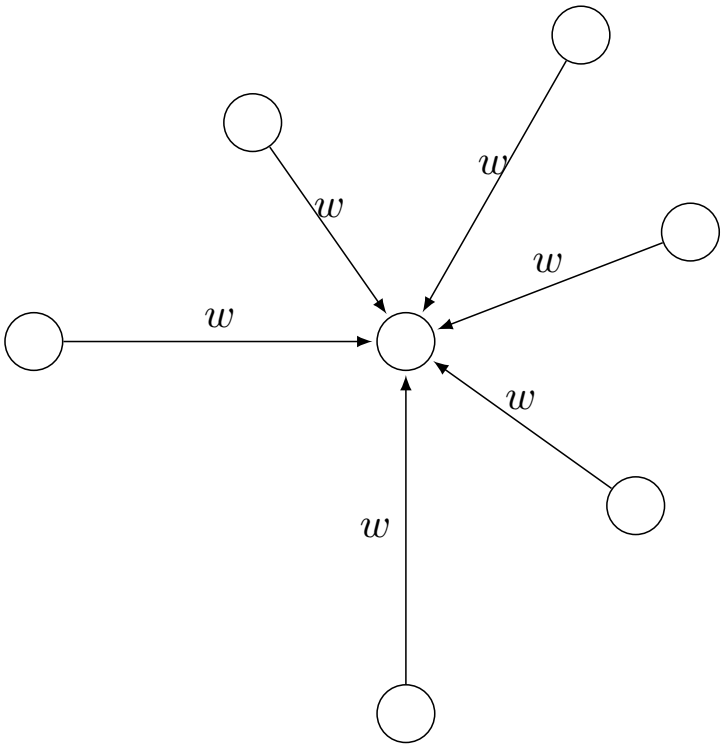

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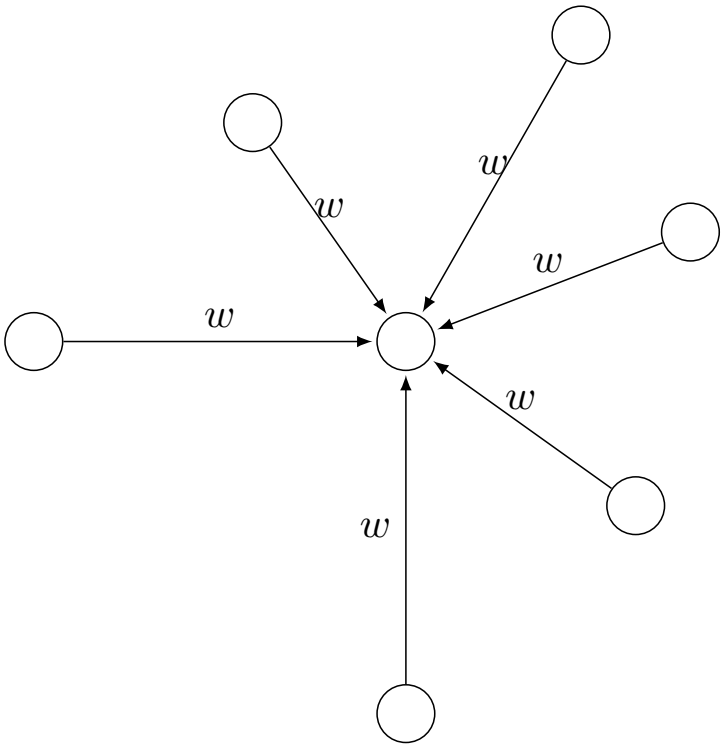
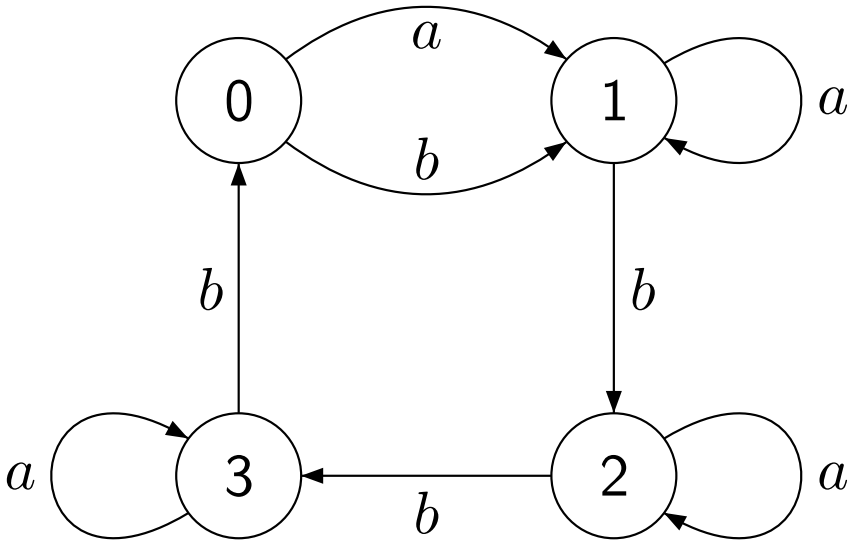


The minimum length of reset words is the reset threshold of $\mathscr{A}$.

Example



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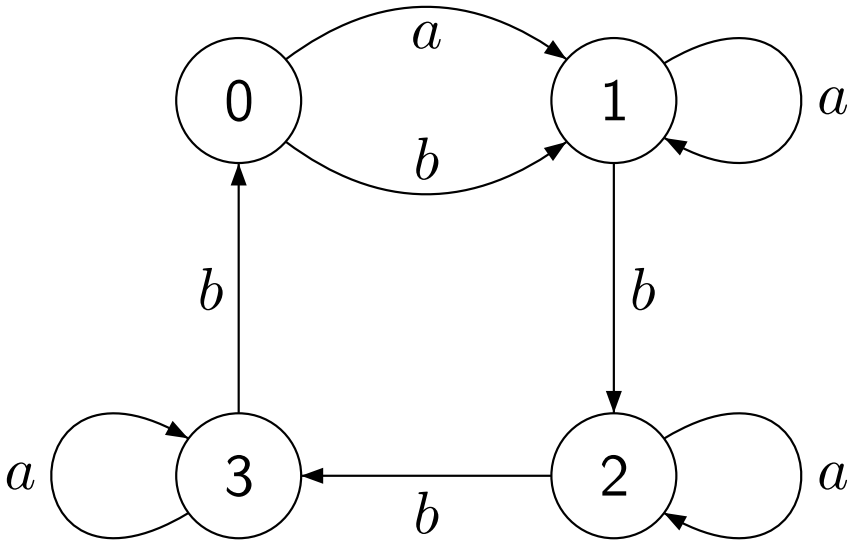

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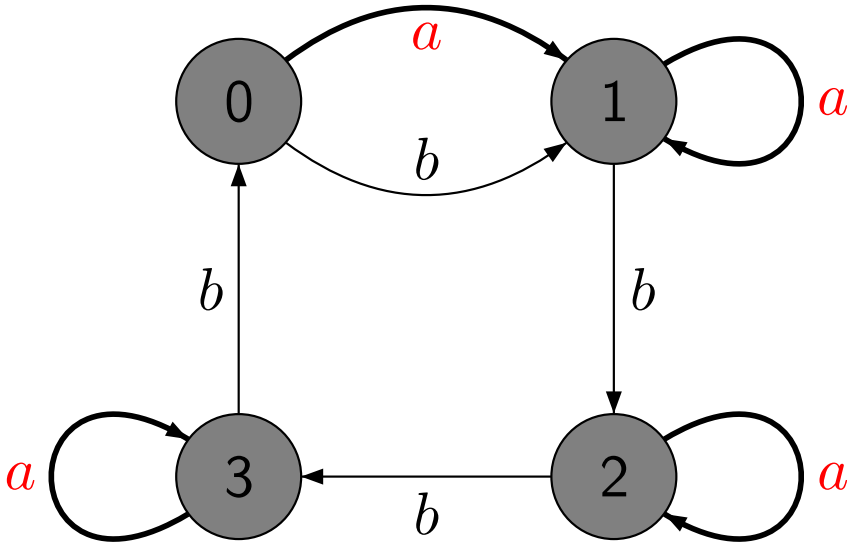

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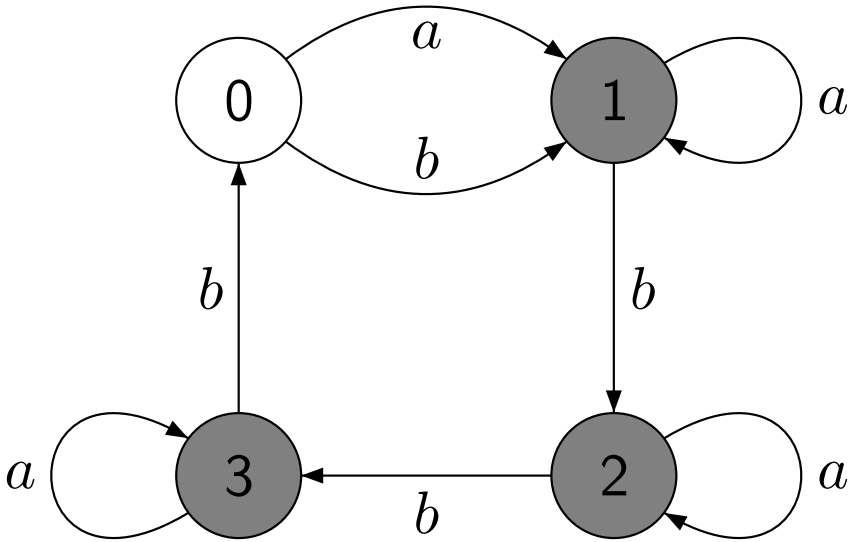

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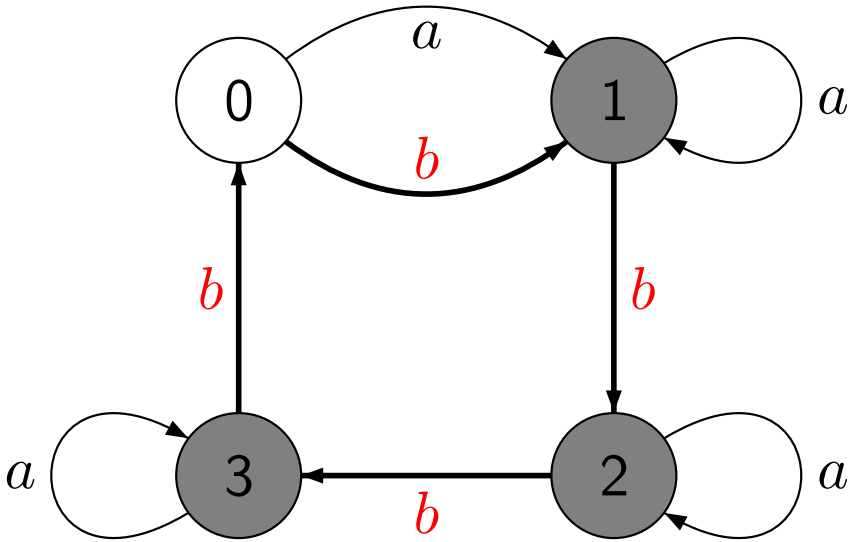

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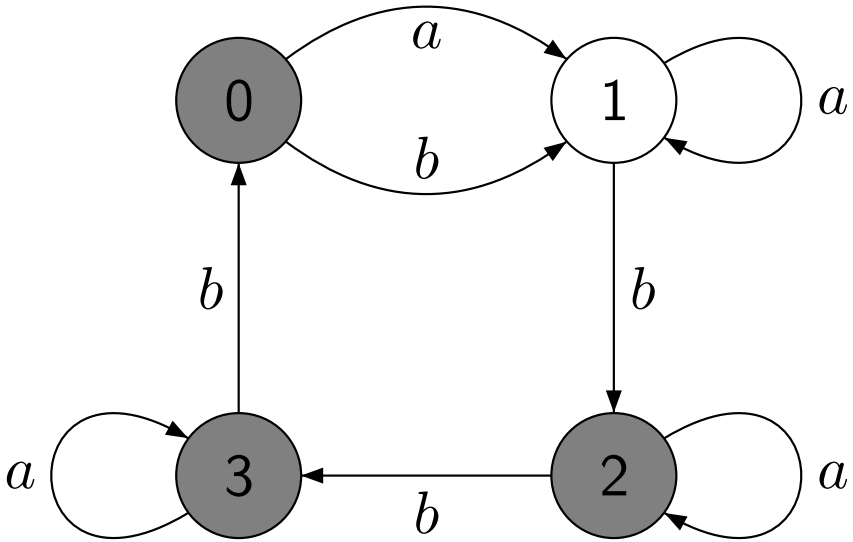

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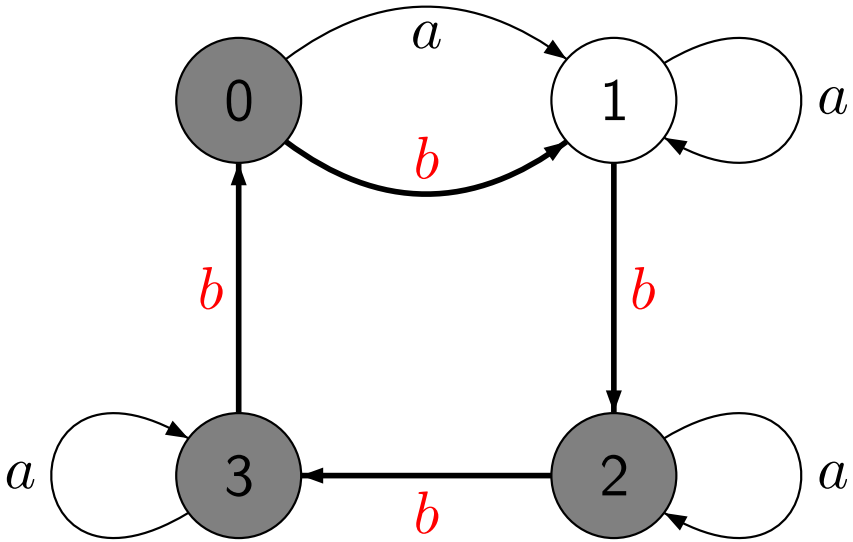

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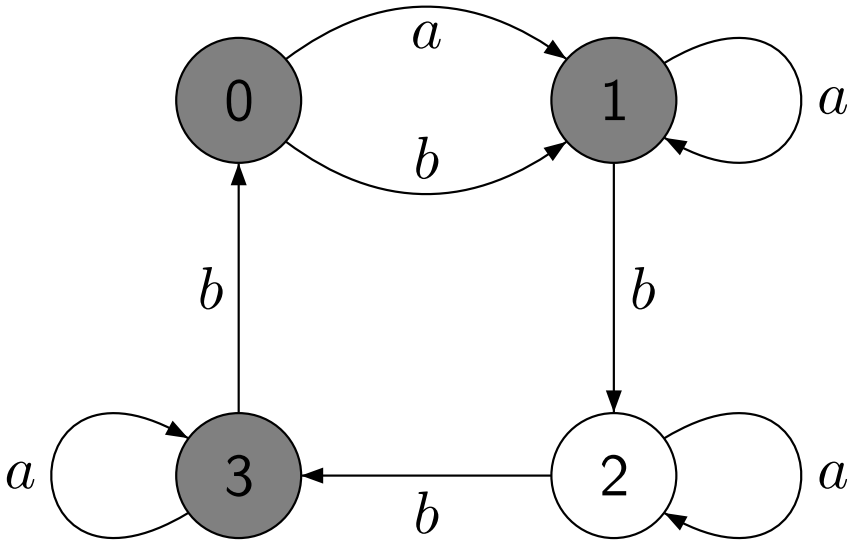

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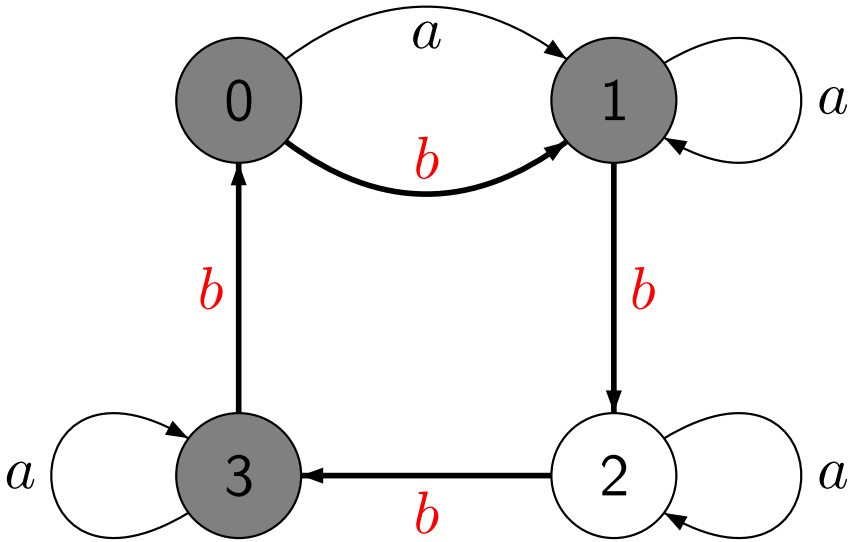

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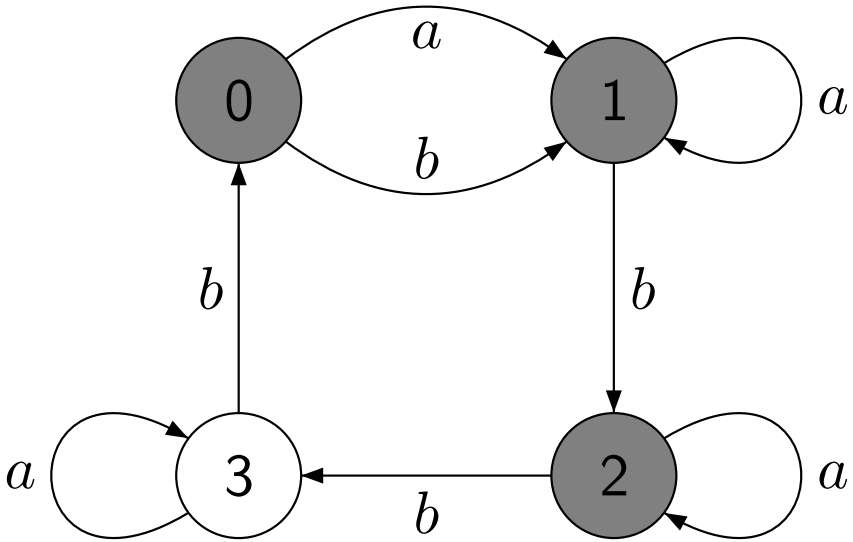

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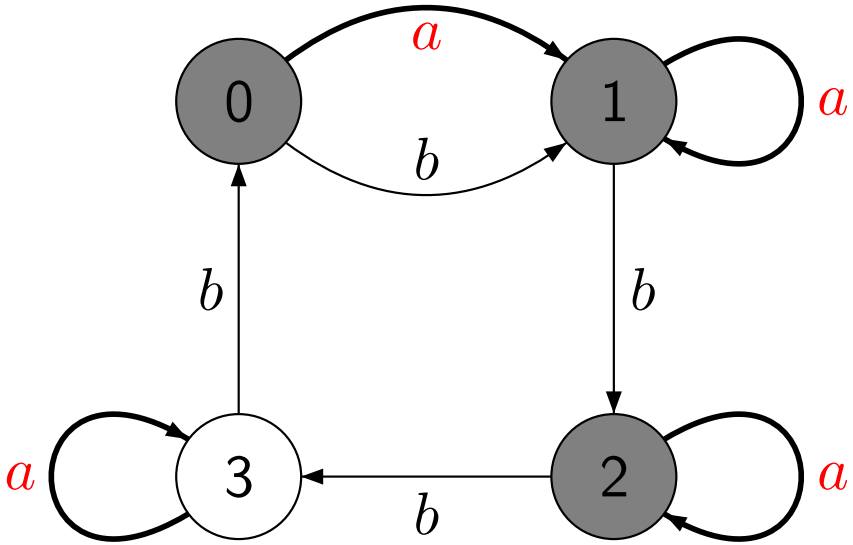

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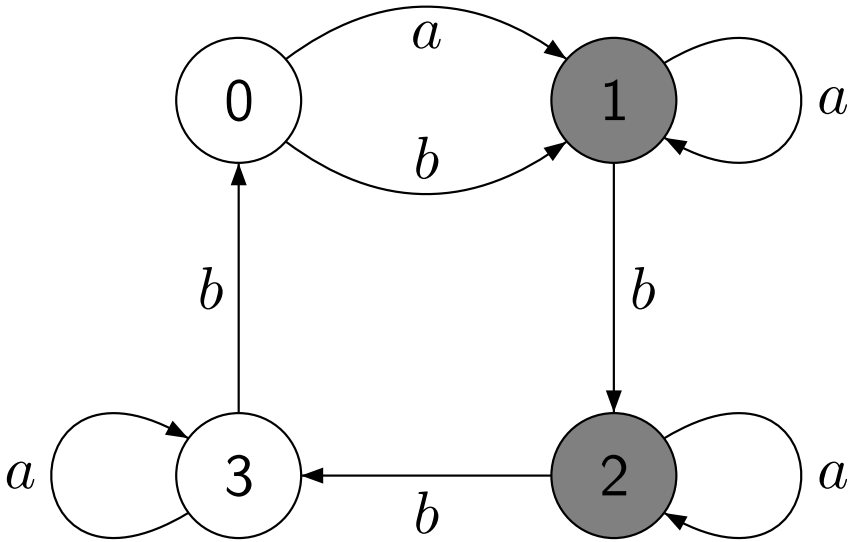

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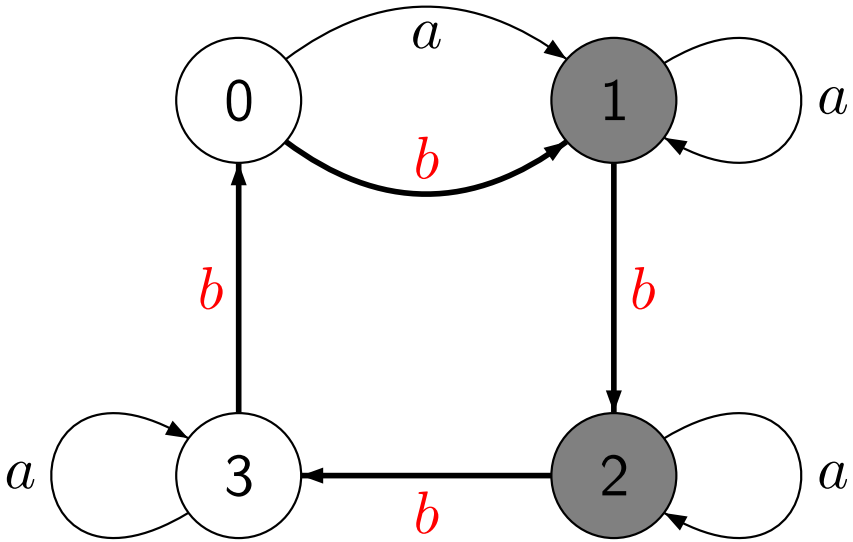

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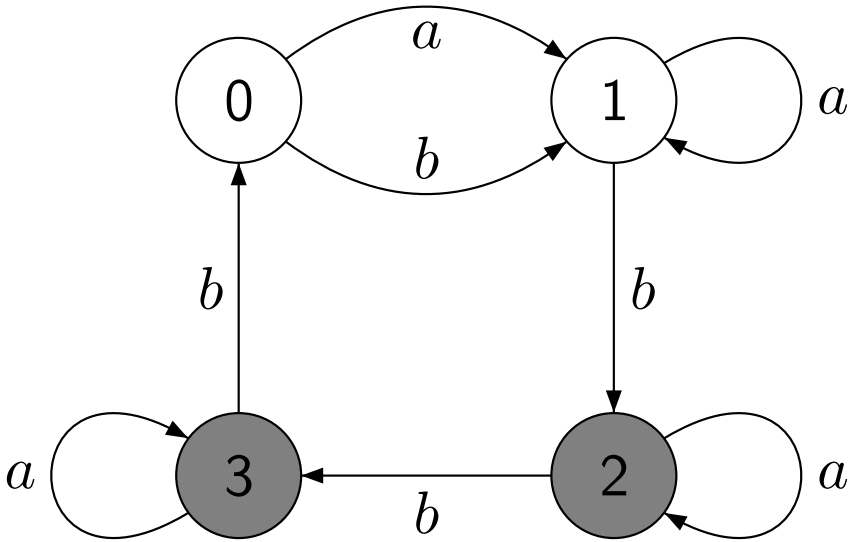

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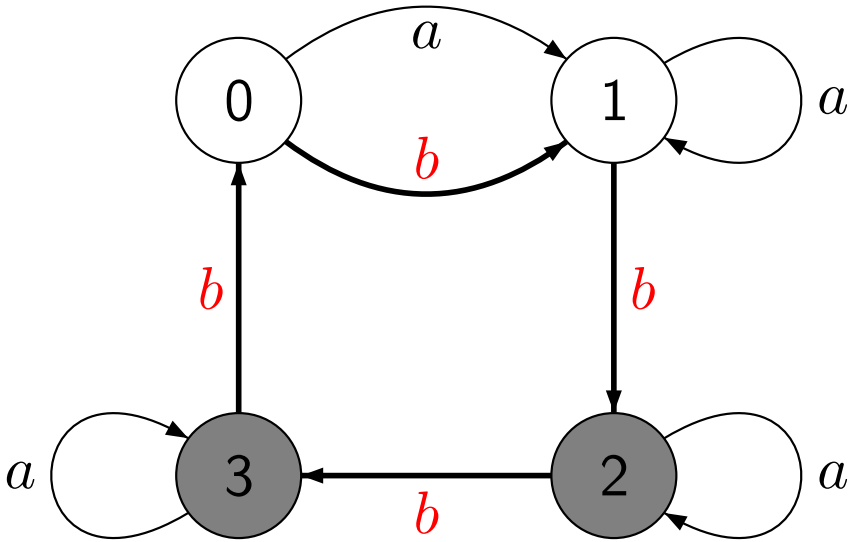

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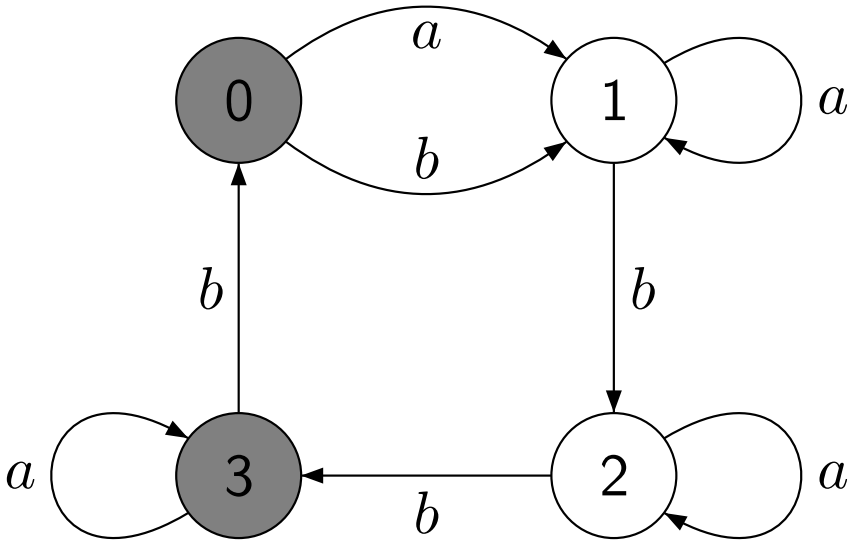

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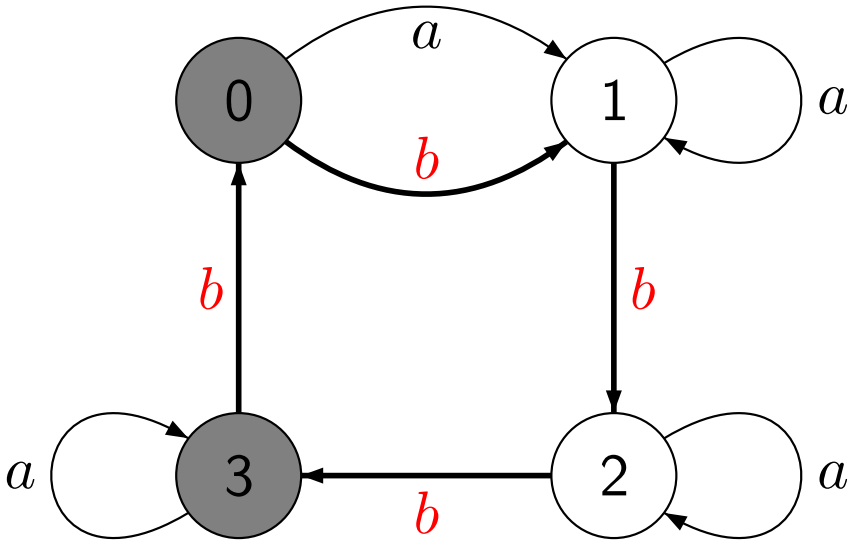

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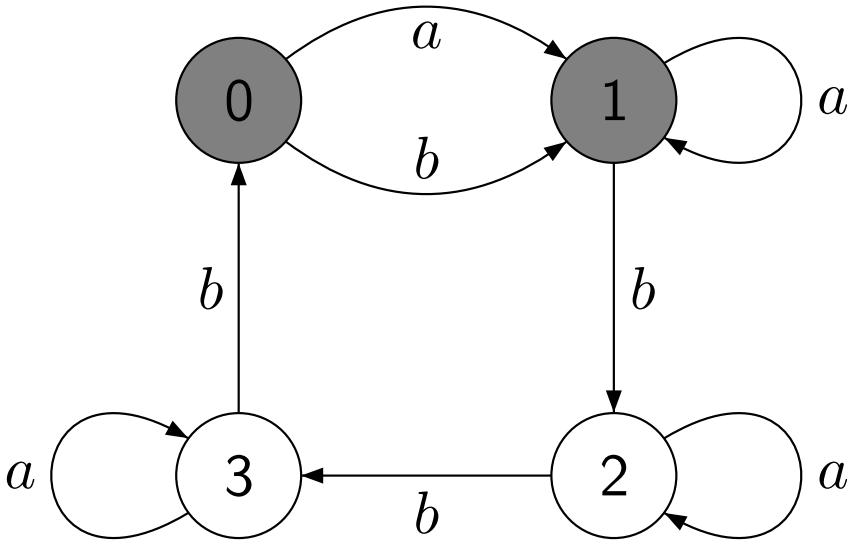

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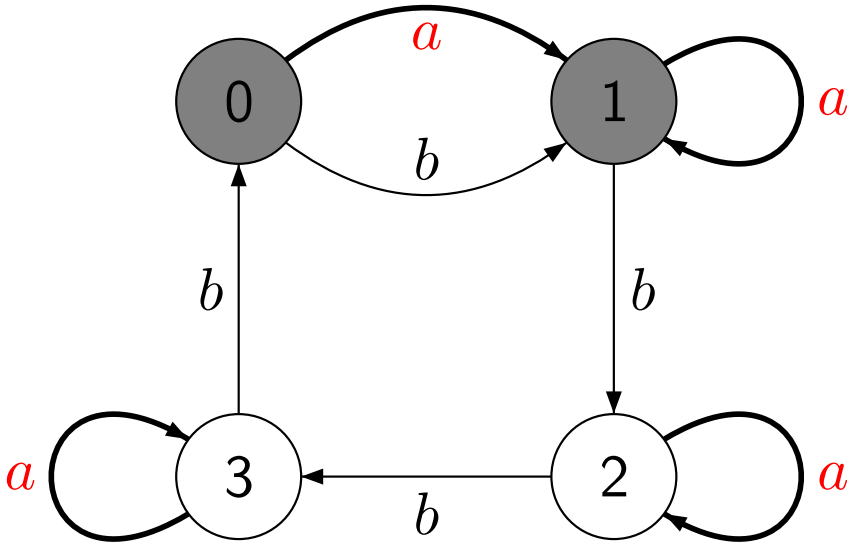

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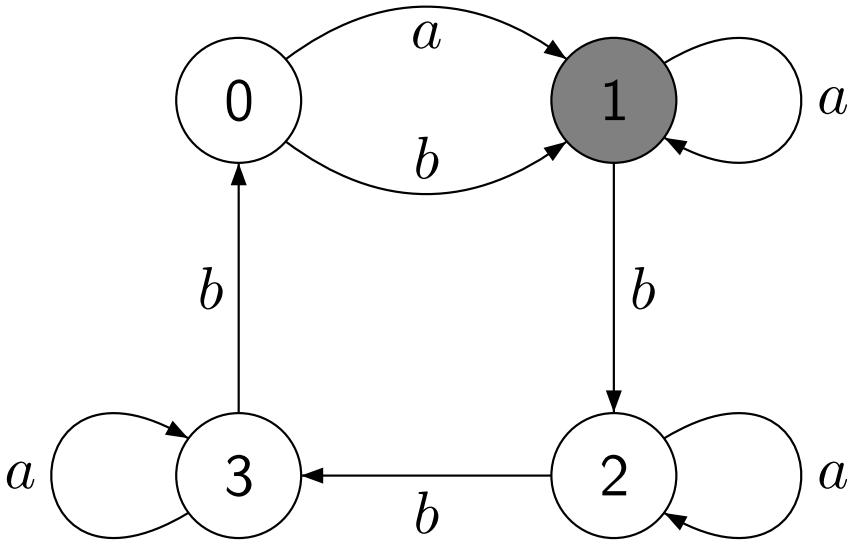

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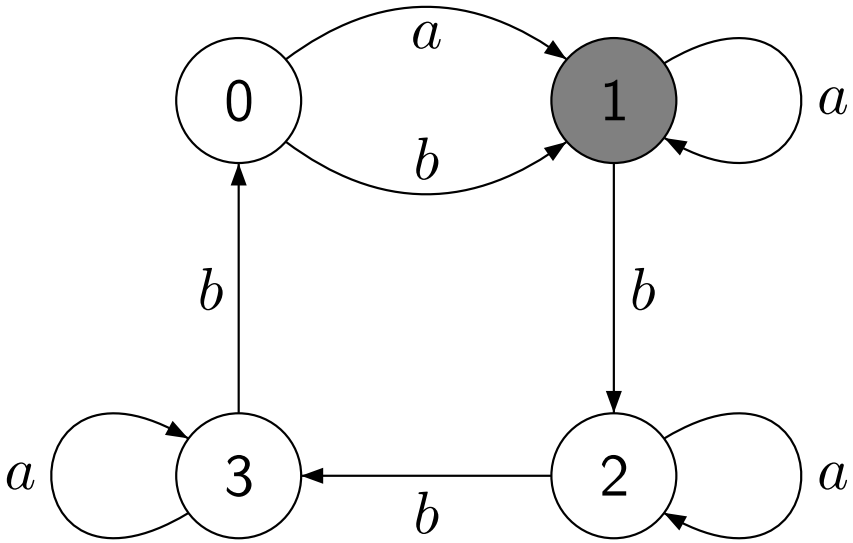

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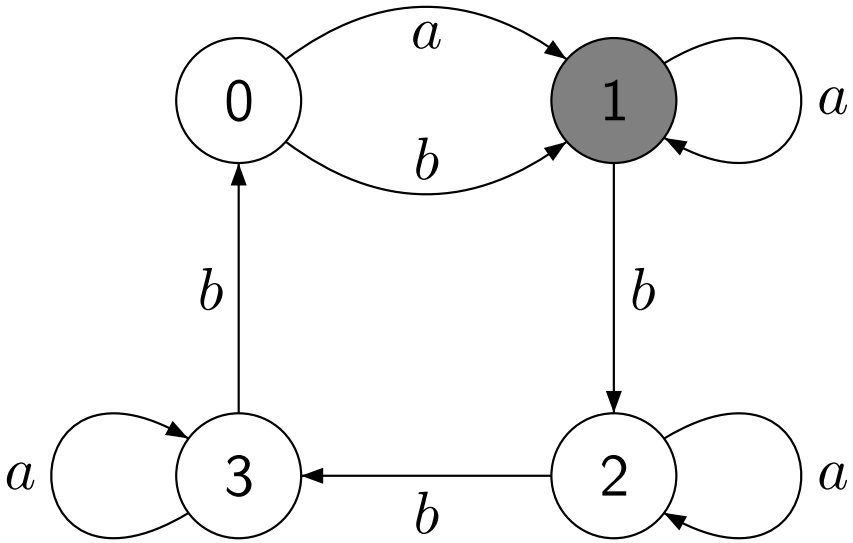

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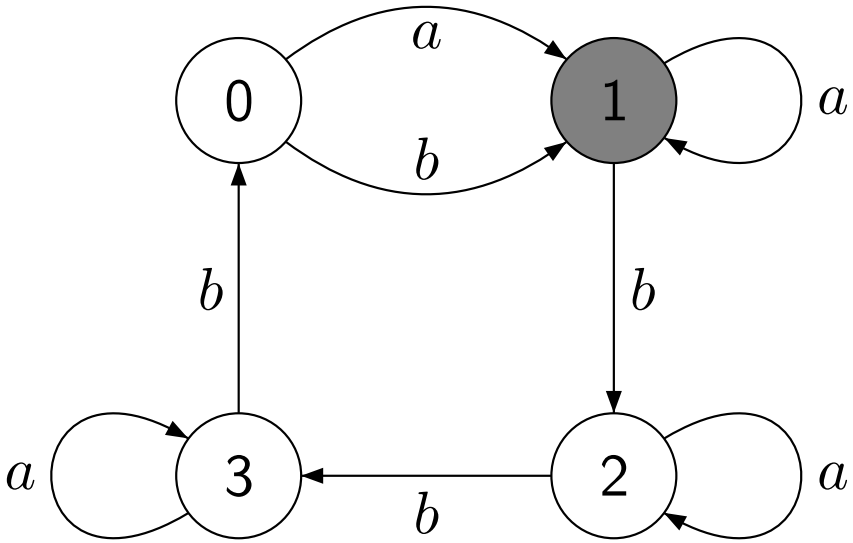

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For each $n > 1$, the automaton $\mathscr{C}_n$ has n states, 2 input letters and reset threshold $(n-1)^2$.

The states of $\mathscr{C}_n$ are the residues modulo n, and the input letters a and b act as follows:

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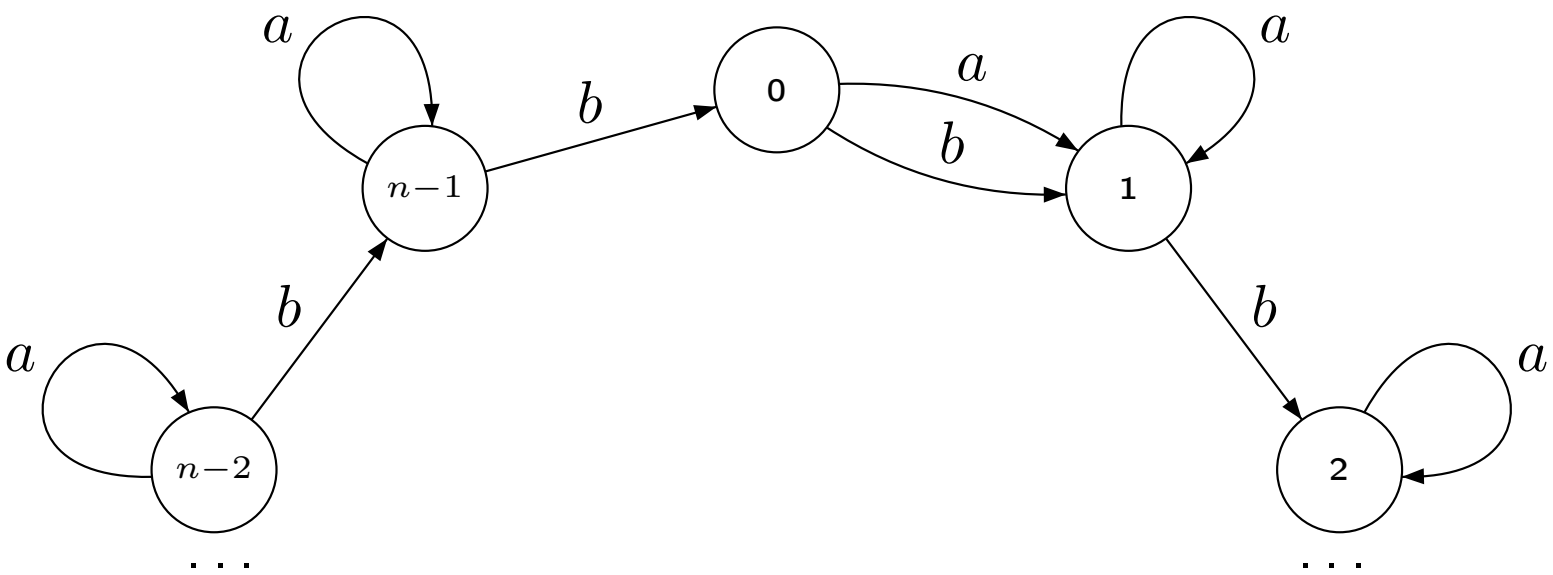

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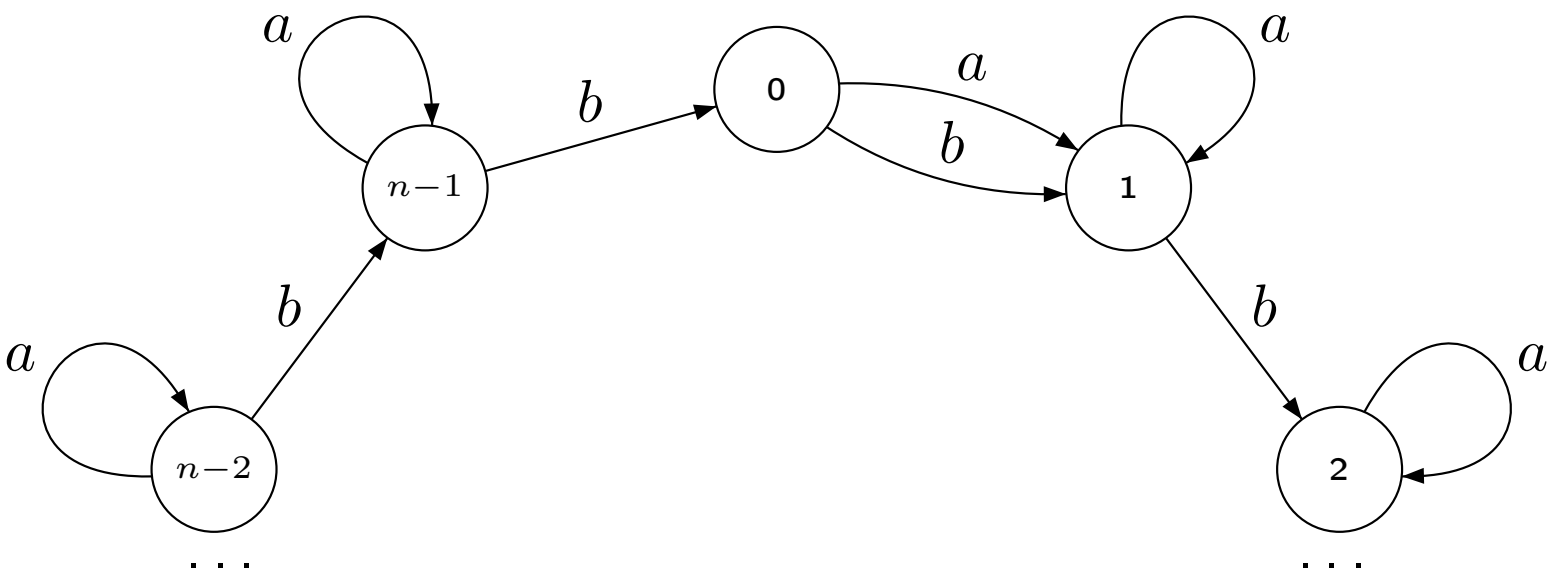

Černý has proved that the shortest reset word for $\mathscr{C}_n$ is $(ab^{n-1})^{n-2}a$ of length $n(n-2)+1 = (n-1)^2$.

Černý conjecture

Define the Černý function $C(n)$ as the maximum reset threshold of all synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, yields the inequality $C(n) \geq (n-1)^2$.

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A small improvement on this bound has been found by Marek Szykuła (published in STACS 2018): his bound is still cubic in n but improves the coefficient $\frac{1}{6} = 0.1666\ldots$ at n^3

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Černý conjecture

Define the Černý function $C(n)$ as the maximum reset threshold of all synchronizing automata with n states. The above property of the series $\{\mathscr{C}_n\}$, yields the inequality $C(n) \geq (n-1)^2$.
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Well, let's speak about synchronization games.

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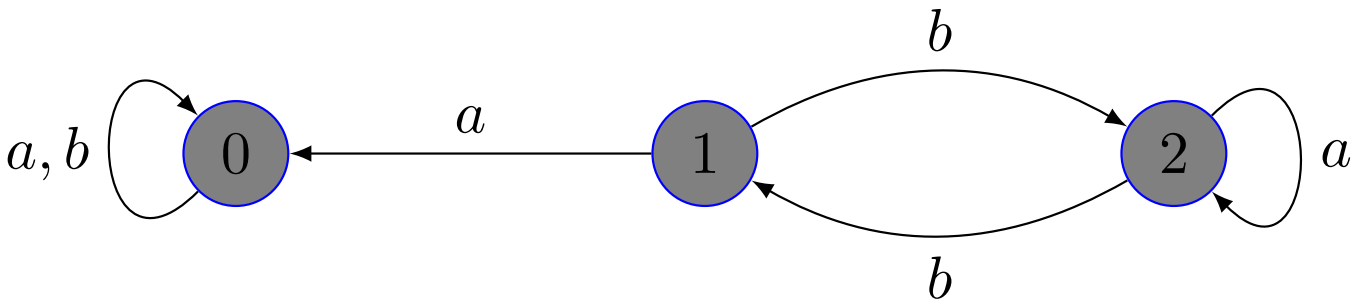
Synchronization game example

To visualize the game on $\mathscr{A} = \langle Q, \Sigma \rangle$, imagine that at the start, each state in Q holds a token.



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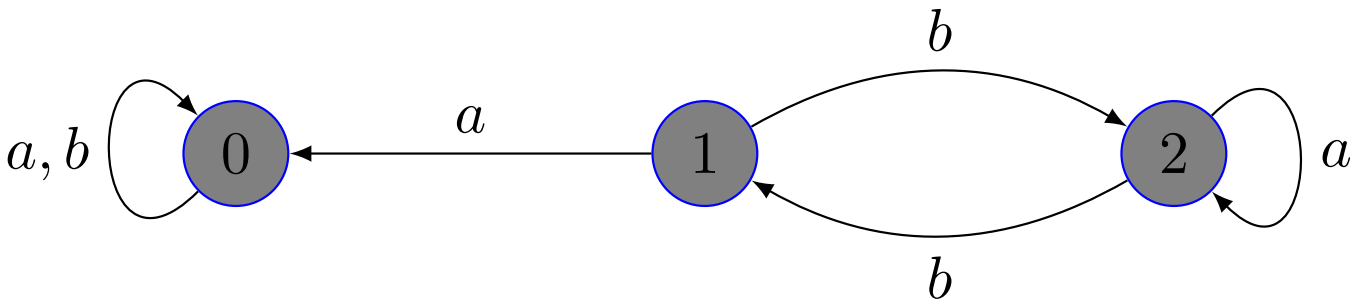

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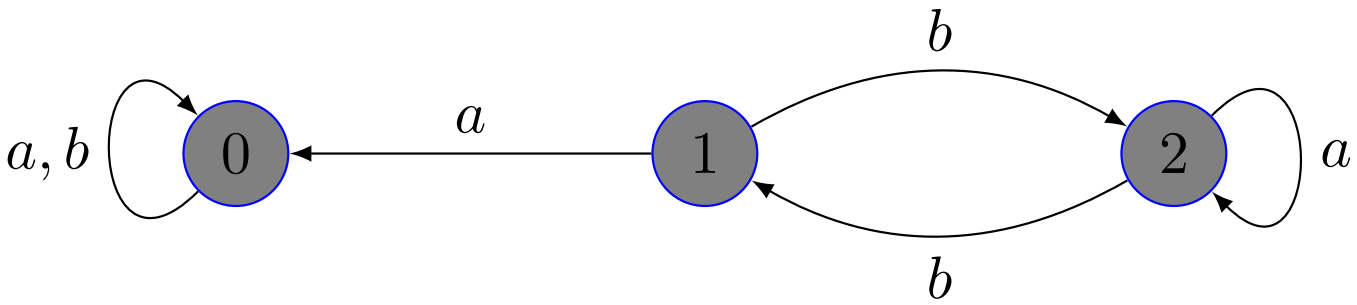
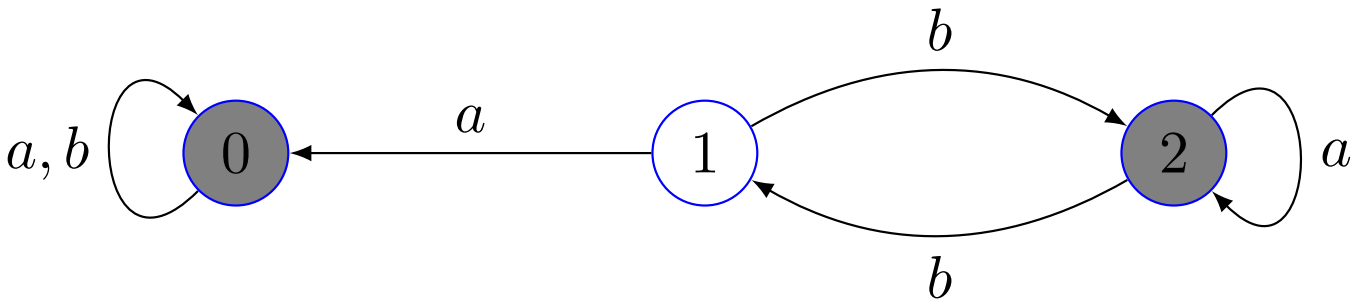
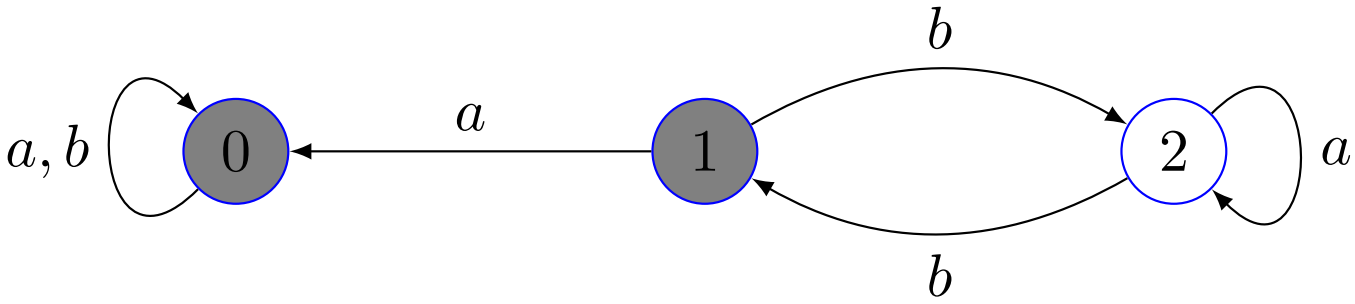


If Alice chooses the input a on her first move, the tokens in 0 and 2 remain due to the loops. The token from 1 moves to 0. Hence, the position after Alice's first move looks as follows:



Synchronization game example, continued

If Bob responds by choosing the input b, the token in 0 remains while the token from 2 moves to 1. Here is the position after Bob's response:



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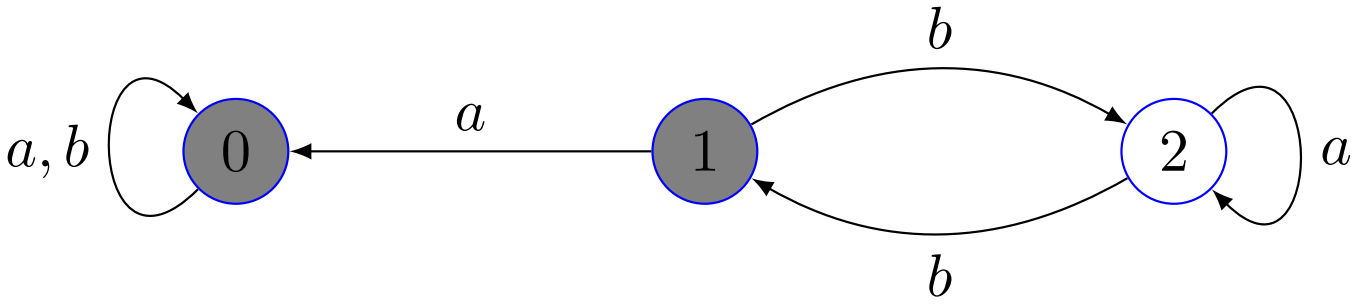
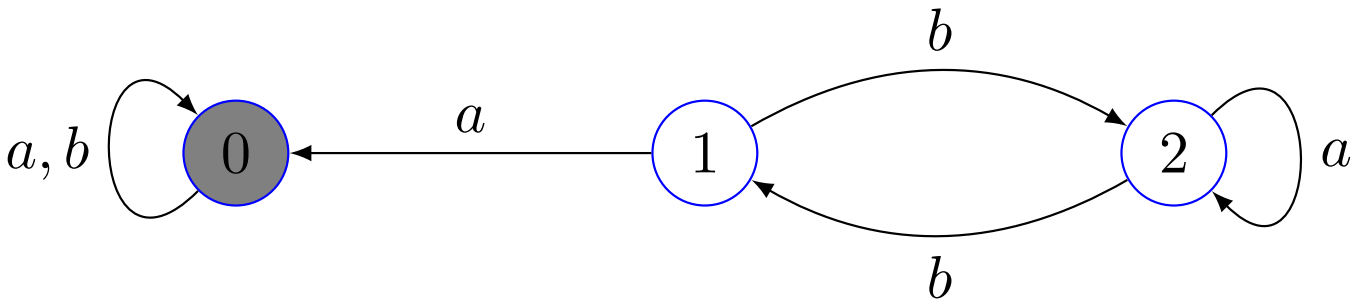


Now choosing a, Alice wins because the token from 1 moves to 0, and we get the position with all tokens in one state:



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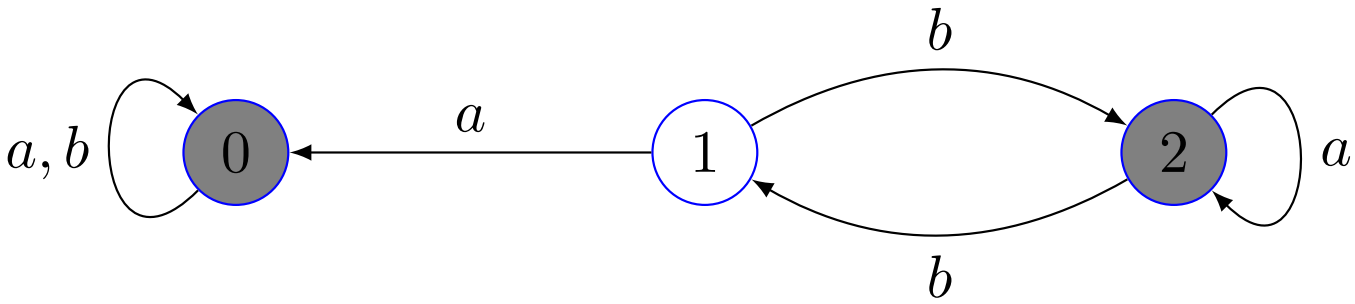

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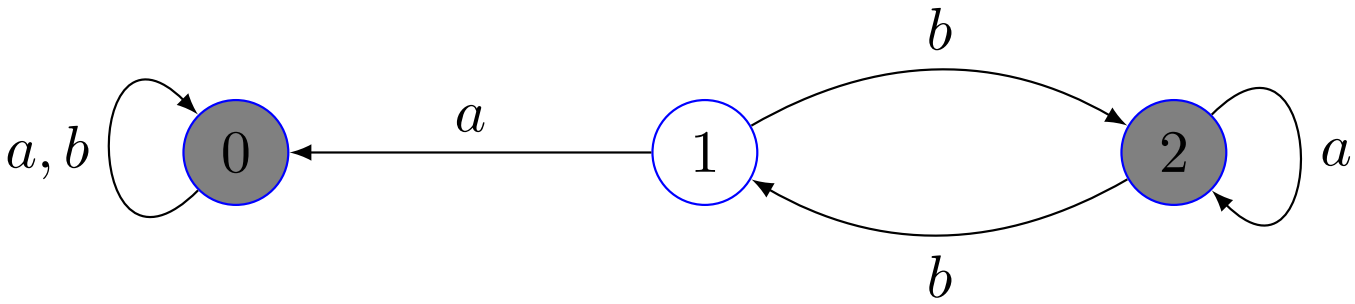

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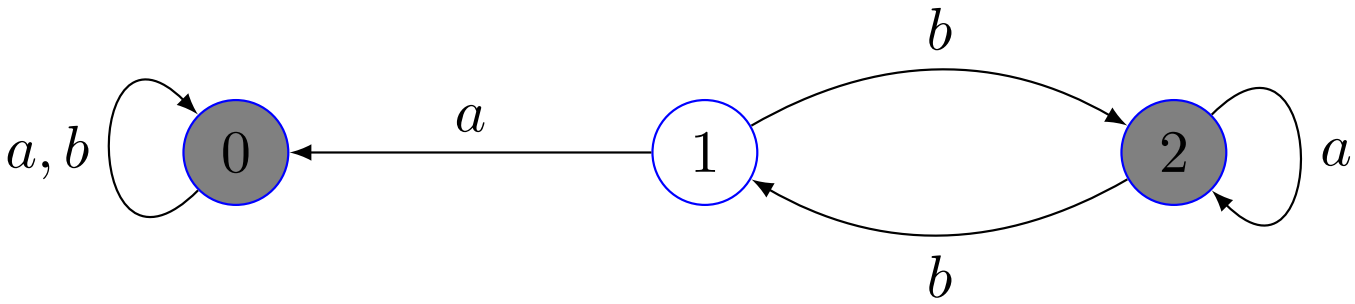

If both players play optimally, the outcome of the game depends solely on the automaton. Our 2012 paper and its follow-ups addressed the problem of classifying automata into those for which Alice has a winning strategy and those for which Bob does.

Adversarial synchronization

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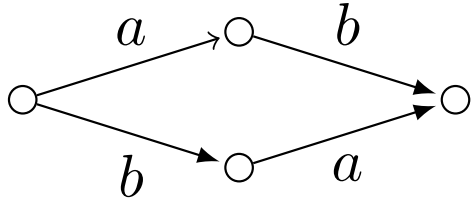
For brevity, let A-automata be automata that admit a winning strategy for Alice under the modified rules. Do A-automata exist?

A-Automata: Simple example

An automaton $\mathscr{A} = \langle Q, \Sigma \rangle$ is said to be commutative if

$qab = qba$ for all $a, b \in \Sigma$ and $q \in Q$:



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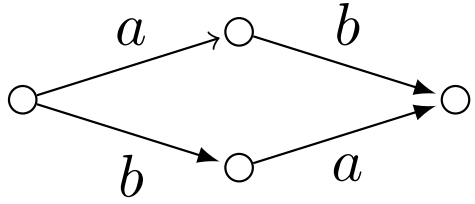

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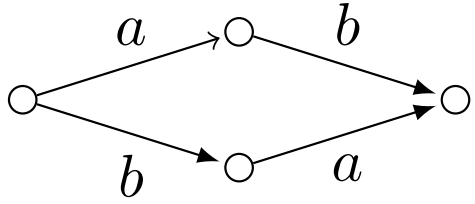

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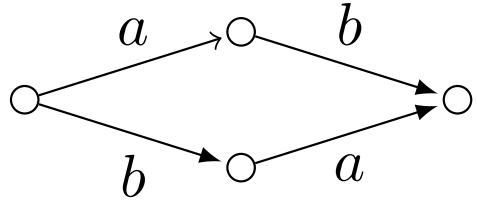

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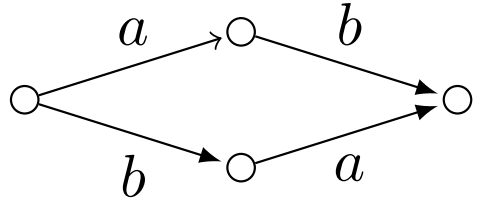

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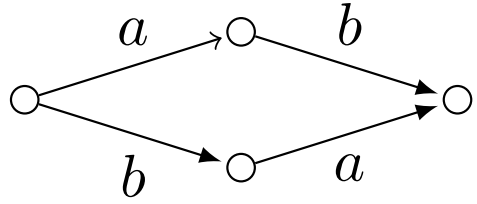

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In fact, the class of A-automata is much more extensive than that of commutative synchronizing automata and includes many types of synchronizing automata studied in the literature.

Reset threshold of A-automata

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Our main result confirms this expectation:

Theorem

The reset threshold of every A-automaton is strictly less than the number of its states.

Proof idea

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In a rough approximation, the idea of the proof is to assign to each position in the synchronization game on an A-automaton $\langle Q, \Sigma \rangle$ the value of a certain parameter with the following properties:

- Alice always has a move that decreases the current value (in an appropriate sense), while no move of Bob can increase it;
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We will now discuss an application of the main result.

Transition monoids and synchronization

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Synchronization is actually a property of transition monoids: $\mathscr{A}$ is synchronizing iff $T(\mathscr{A})$ contains a constant transformation.

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The transition monoid $T(\mathscr{A})$ of $\mathscr{A}$ is the submonoid of the monoid of all transformations on Q generated by the set $\{\tau_a \mid a \in \Sigma\}$.

$T(\mathscr{A})$ can alternatively be defined as the monoid of all maps $\tau_w : Q \to Q$ defined by $q \mapsto qw$ where $w \in \Sigma^*$.

Synchronization is actually a property of transition monoids: $\mathscr{A}$ is synchronizing iff $T(\mathscr{A})$ contains a constant transformation.

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Still, certain properties of transition monoids (e.g. commutativity) may force a synchronizing automaton to be an A-automaton.

DS-automata

The set $\mathbf{DS}$ of all finite monoids with regular $\mathfrak{D}$-classes being subsemigroups is a variety playing a distinguished role in the algebraic theory of regular languages and computational complexity.

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Recall: Green's relations on a monoid M

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A DFA whose transition monoid lies in $\mathbf{DS}$ is a DS-automaton.

Proposition (Fernau, Haase, Hoffmann, MV, arXiv:2512.11007)

Each synchronizing DS-automaton is an A-automaton.

Application

Corollary

The reset threshold of every synchronizing DS-automaton is strictly less than the number of its states.

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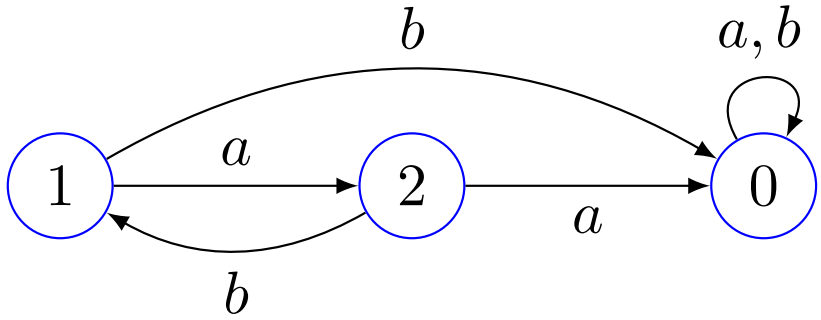
An A-automaton need not be a DS-automaton. It is known that $\mathbf{DS}$ excludes the Brandt monoid B_2^1 consisting of the following six matrices, with standard matrix multiplication:

$$\begin{pmatrix}1&0\\0&0\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}, \begin{pmatrix}0&0\\0&1\end{pmatrix}, \begin{pmatrix}0&0\\0&0\end{pmatrix}, \begin{pmatrix}1&0\\0&1\end{pmatrix}.$$

An A-automaton beyond $\mathbf{DS}$

B_2^1 is the transition monoid of



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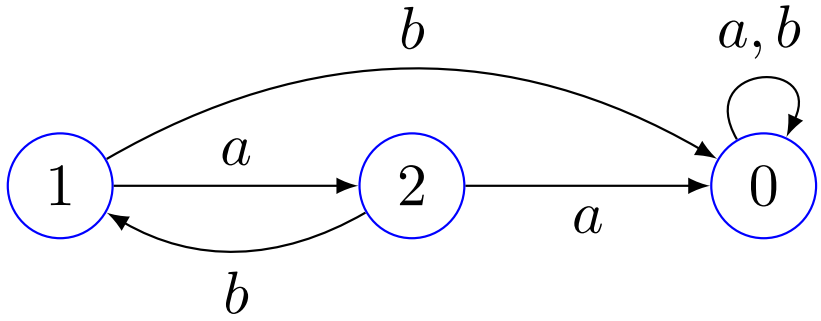
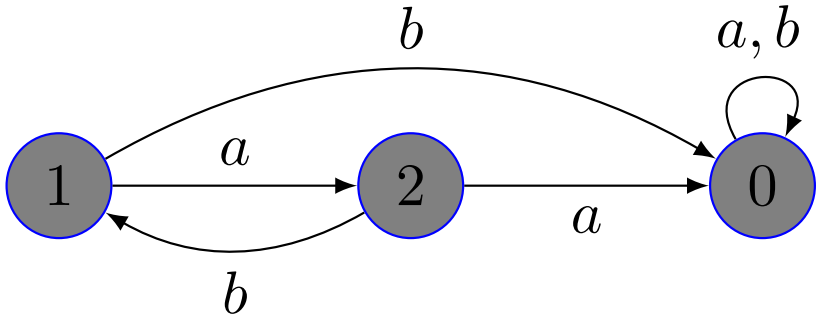


Alice wins on this automaton.



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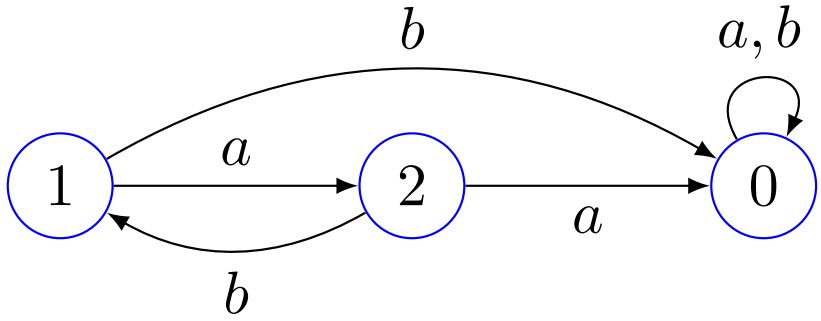

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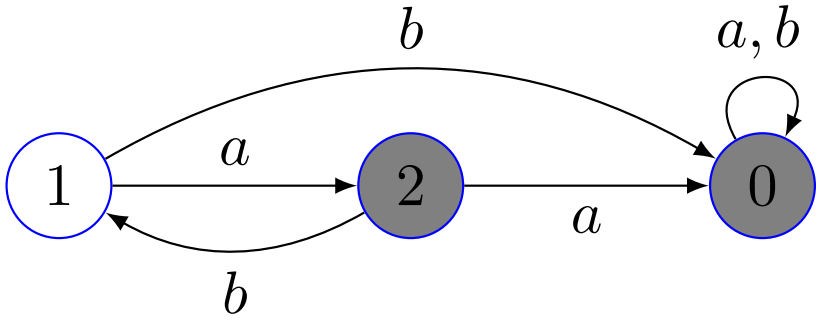

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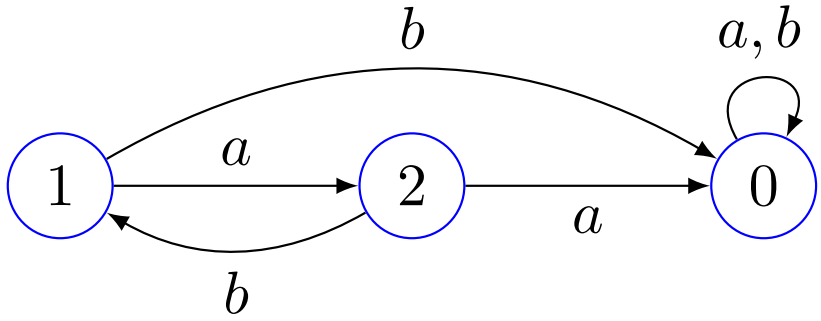

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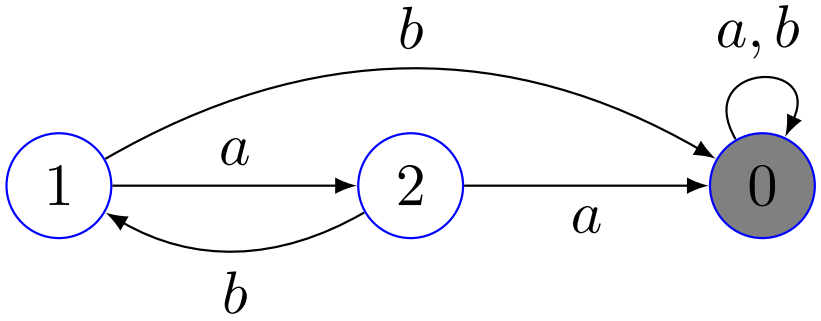

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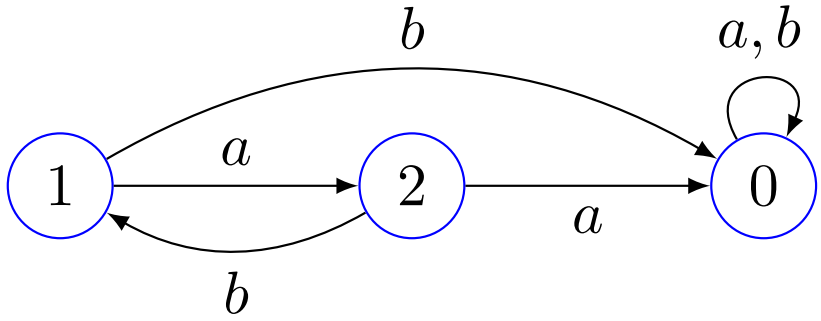

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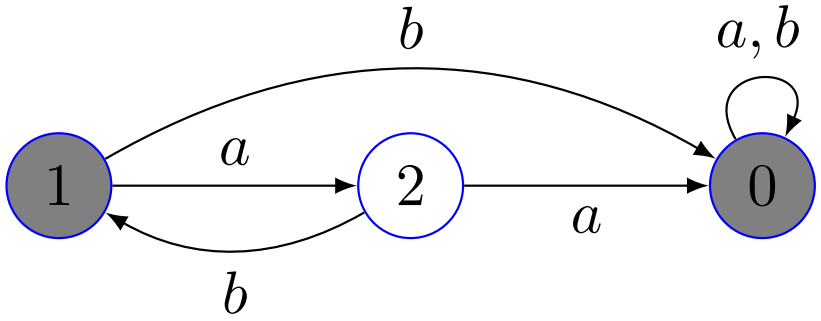

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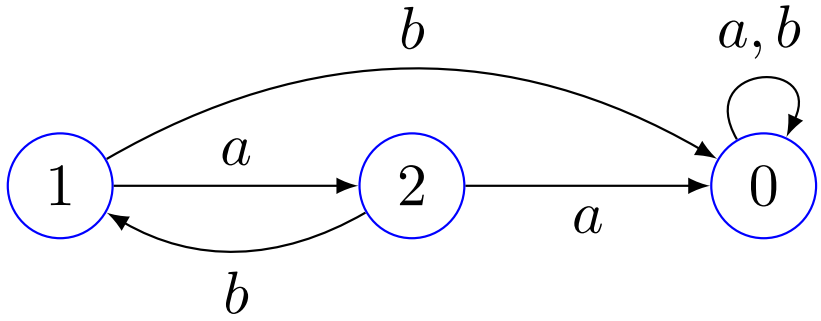

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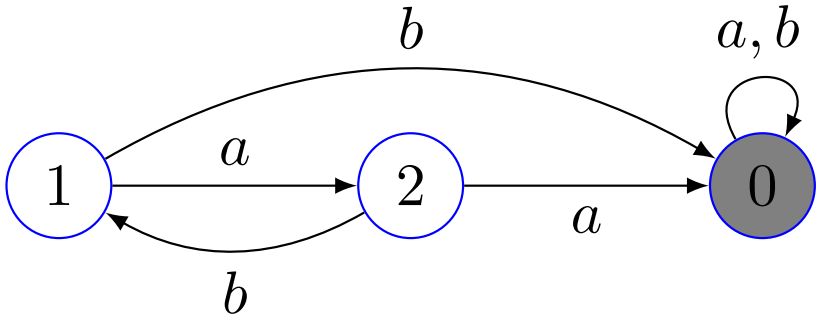

She can start by choosing a. If Bob responds by skipping his turn or by choosing a word that ends with a, Alice wins by choosing a. If Bob responds by choosing some word that ends with b, then Alice wins by choosing b.

Restricting Bob's capacity

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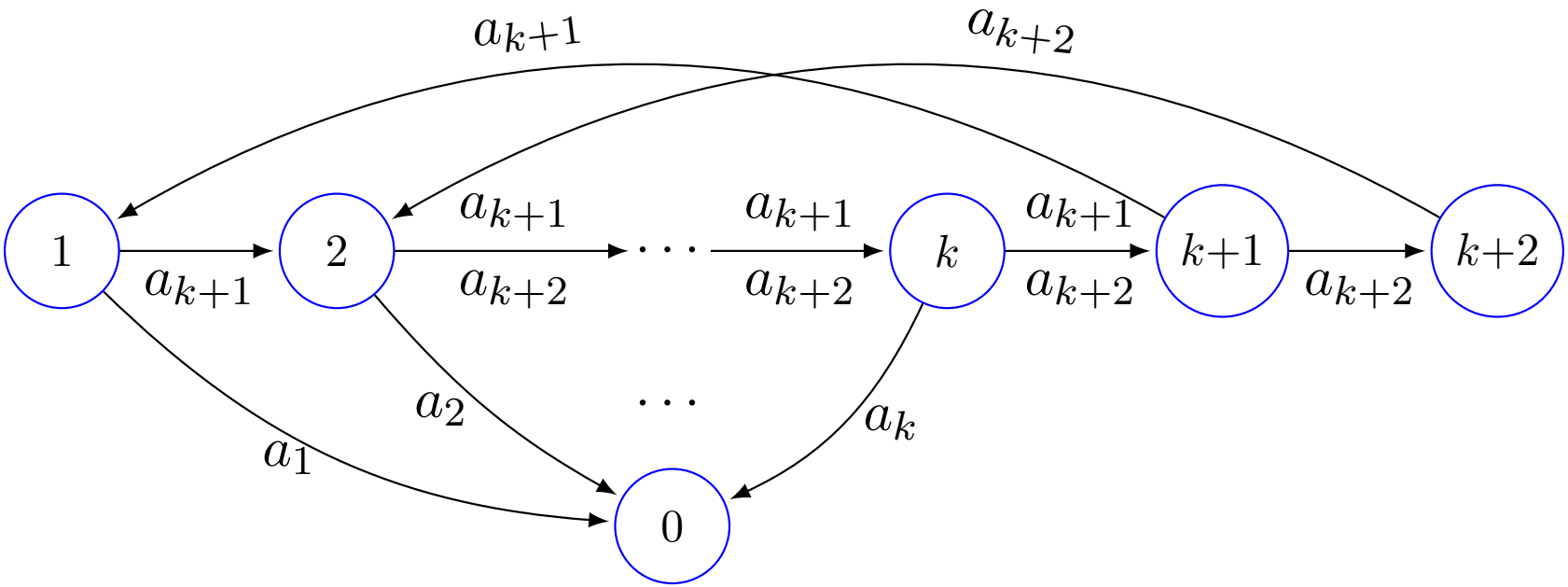

Enhancing Alice's capacity

What happens if Alice, like Bob, is allowed to play words rather than single letter? If Alice may play arbitrary words, the game trivializes as Alice instantly wins on any synchronizing automaton by spelling out a reset word. The variant in which Alice's moves restricted to words of length at most $m \in \mathbb{Z}_+$ is more interesting.

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The Černý conjecture holds for n-state DFAs on which Alice wins by playing words of length at most n.

Algorithmic issues

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The DFAs $\mathscr{A}$ and $\mathscr{A}^{[2]}$ have the same reset words. From this, it is easy to see that the DFAs are equivalent with respect to adversarial synchronization.

Algorithmic issues, continued

A-automata with a sink admit a transparent characterization:

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A DFA $\langle Q, \Sigma \rangle$ with a sink s is an A-automaton iff for every $q \neq \mathrm{s}$ there exists a letter $a \in \Sigma$ such that the states q and qa belong to distinct strongly connected component of $\langle Q, \Sigma \rangle$.

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For the version with Bob's moves restricted to words of length less than $k \in \mathbb{Z}_+$, a polynomial time algorithm also exists but it and its justification are much more involved.

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- Thank you for your attention!