Automata-based Verification of Quantum Circuits

Ondřej Lengál

Brno University of Technology, Czech Republic

joint work with
Parosh Aziz Abdulla, Yu-Fang Chen, Yo-Ga Chen, Kai-Min Chung, Michal Hečko, Lukáš Holík, Min-Hsiu Hsieh, Wei-Jia Huang, Jyun-Ao Lin, Fang-Yi Lo, Ramanathan S. Thinniyam, Wei-Lun Tsai, Di-De Yen

One FLAT World Seminar (20 May 2026)

Why Quantum Computation?

Quantum computation:

- first proposed by Feynman (1982)
- promises to efficiently solve some problems we don't know how to efficiently solve classically

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- $\leadsto$ we need to be prepared (computer-aided analysis)
- **FUN** and cool community!

Outline

Short Quantum Introduction

Bit vs. Qubit

A bit

$x = 0 \quad \text{or} \quad x = 1$

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A qubit

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$$\alpha, \beta \in \mathbb{C}, \quad \text{s.t.} |\alpha|^2 + |\beta|^2 = 1$$

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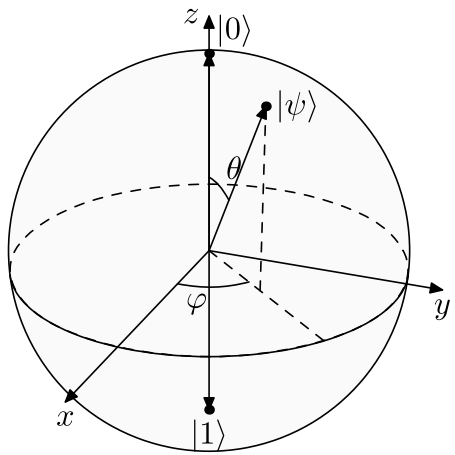
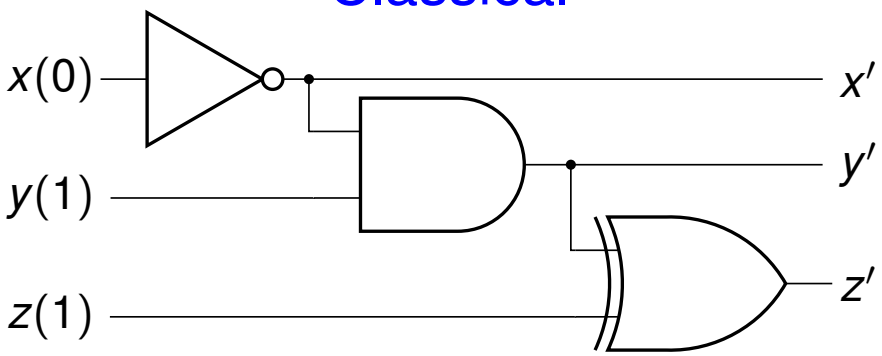


Classical vs. Quantum Circuits — State

Classical



x'	y'	z'	χ
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	**1**	**0**	**1**
1	1	1	0

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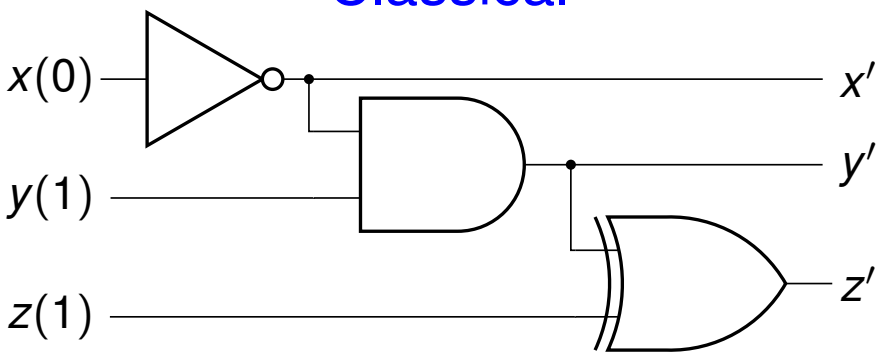

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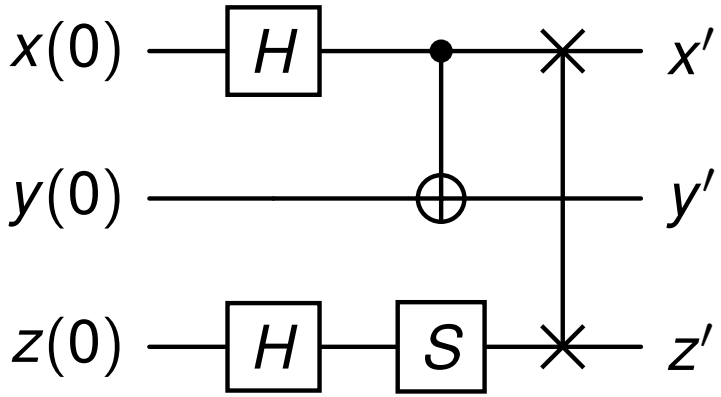

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x'	y'	z'	amp
0	**0**	**0**	**½**
0	0	1	0
0	1	0	0
0	**1**	**1**	**½**
1	**0**	**0**	**½ i**
1	0	1	0
1	1	0	0
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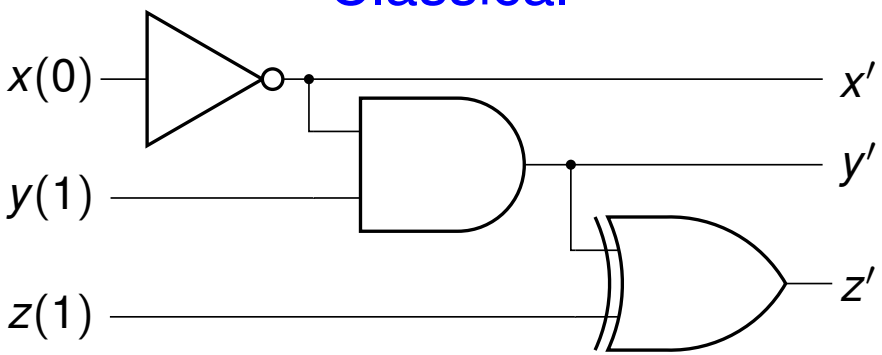
$amp(\vec{x}) \in \mathbb{C}$

$\Pr(\vec{x}) = |x|^2$

Classical vs. Quantum Circuits — Gates

Classical



A gate is a truth table

a	b	$a \oplus b$
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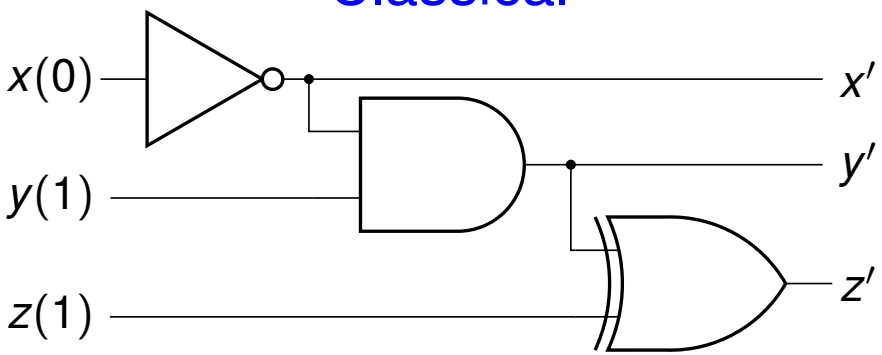

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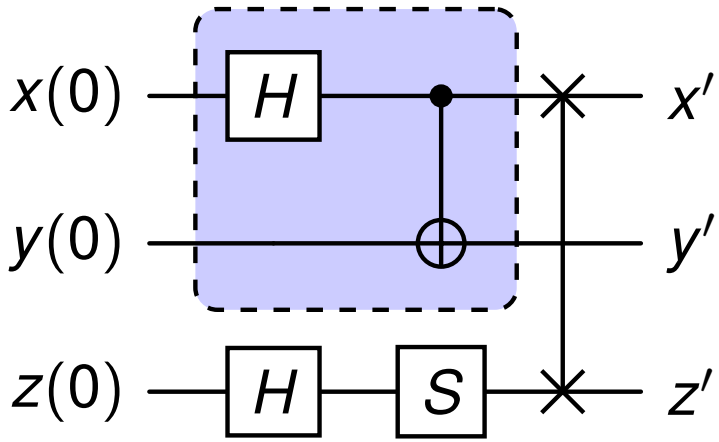
unitary matrix:

- conjugate transpose $U^\dagger = U^{-1}$

Quantum



A gate is a unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

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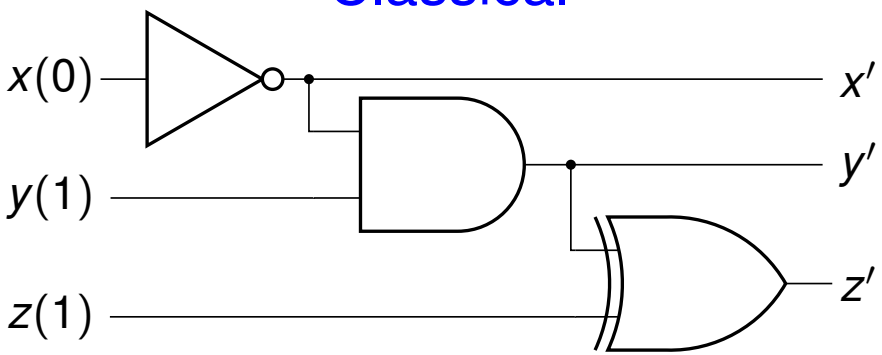

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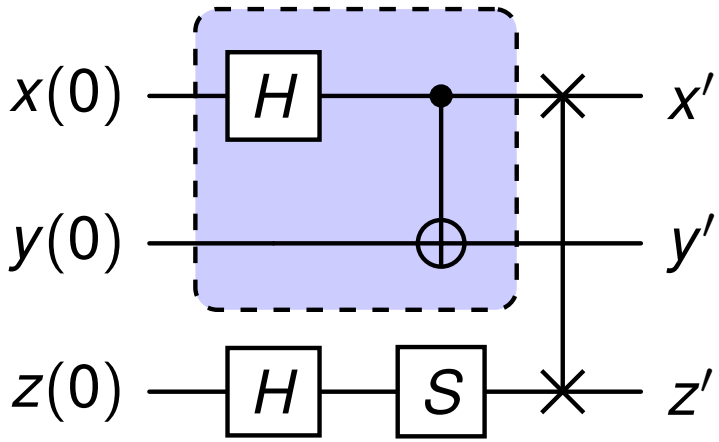

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unitary matrix:

- conjugate transpose $U^\dagger = U^{-1}$
- $\rightsquigarrow$ reversibility, norm preservation, no-cloning theorem, ...

Short Quantum Circuit Example — Hadamard gate

$$H = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$$

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Verification of Quantum Circuits

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Main approaches:

1. state vector simulation (strong: #**P**-complete)
2. equivalence checking (**QMA**-complete)
   - **QMA** = *Quantum Merlin Arthur*; the so-called “quantum NP”
3. (pre/post-condition) verification

Verification of Classical Programs

Verification of classical programs:

- (pre/post-condition based, a.k.a. Floyd-Hoare style)

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- *Pre* and *Post* denote sets of program states

Meaning:

- If S is executed from a state from *Pre*
- and the execution of S terminates,
- then the program state after S terminates is in *Post*.

Verification of Quantum Circuits

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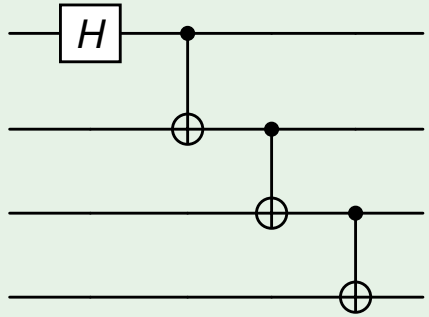
Meaning:

- If *C* is executed from a quantum state from *Pre*
- then the quantum state after *C* terminates is in *Post*.
- (termination is implicit)

Verification of Quantum Circuits

Example (GHZ)



- GHZ (Greenberger–Horne–Zeilinger state)
  - generalization of Bell's state from 2 to arbitrary many qubits
  - maximally entangled qubits

$$|0000\rangle \xrightarrow{GHZ} \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}$$

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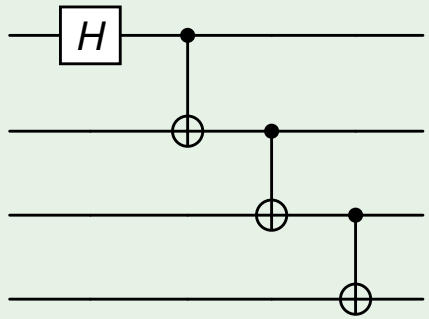

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Verification of Quantum Circuits

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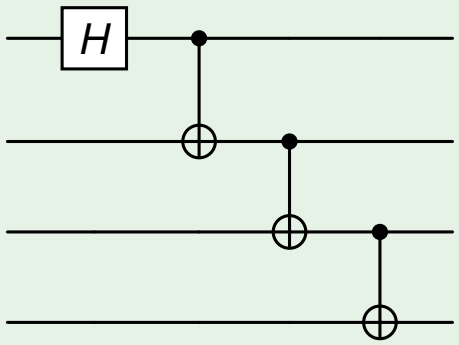
$\{|w\rangle : w \in \{0,1\}^4\}$



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Pre Circuit *Post*

$$Pre = \{|0000\rangle, |0001\rangle, \ldots, |1111\rangle\}$$

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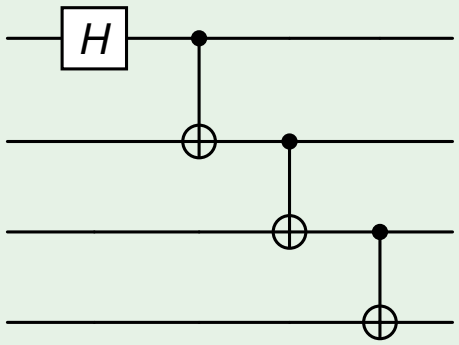

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How to efficiently represent sets of quantum states *Pre* and *Post*?

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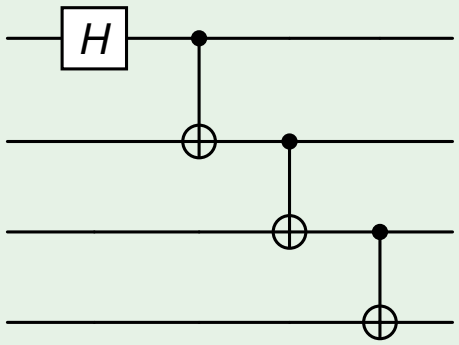

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- naively $\rightsquigarrow$ double exponential size

Quantum States are Trees

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. . . and quantum gates are tree operations

Quantum States are Trees

x	y	z	amp
0	**0**	**0**	**½**
0	0	1	0
0	1	0	0
0	**1**	**1**	**½**
1	**0**	**0**	**½** i
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$\Rightarrow$

½	0	0	½	½i	0	0	½i

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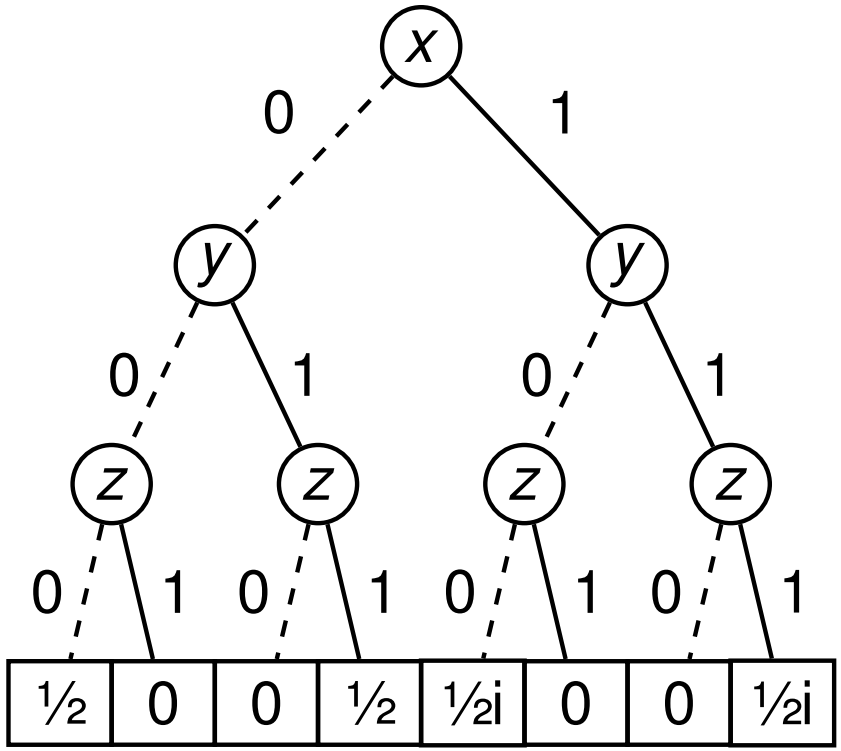

- perfect tree of height n (the number of qubits) $\rightsquigarrow 2^n$ leaves

Quantum Gates are Tree Operations

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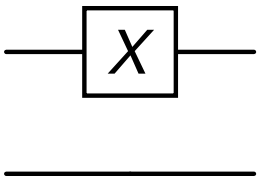
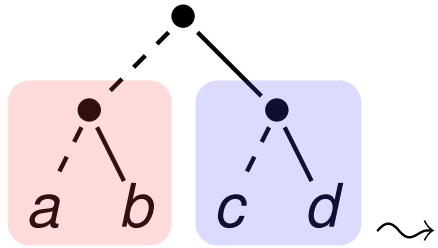


$$X_1 = \overbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}^{X} \otimes \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}^{I}$$



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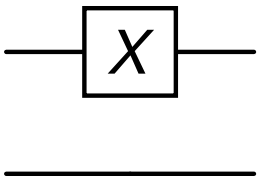

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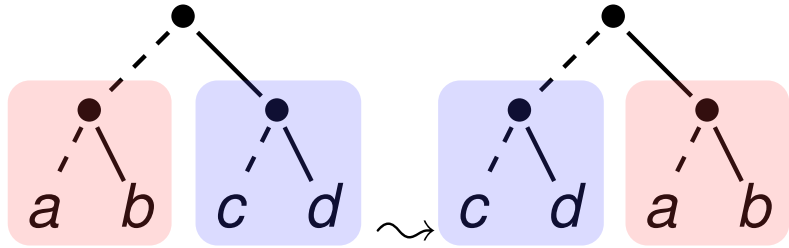

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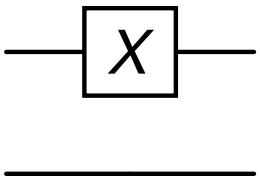

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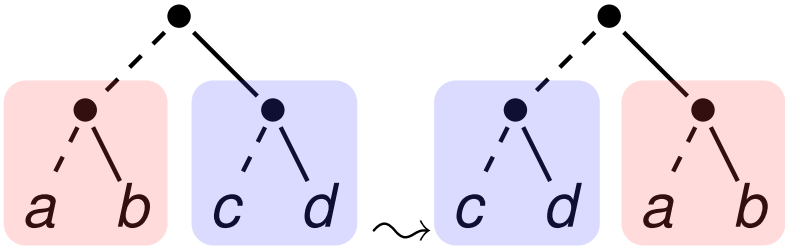

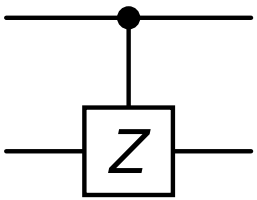

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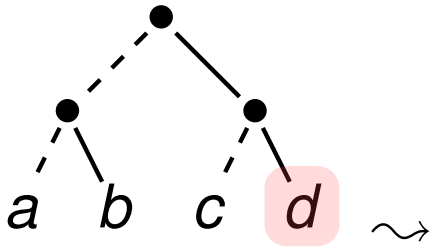
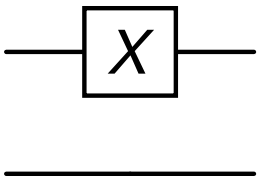


Quantum Gates are Tree Operations



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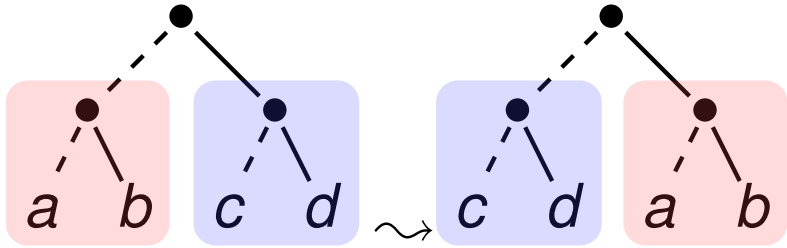

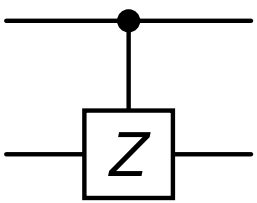

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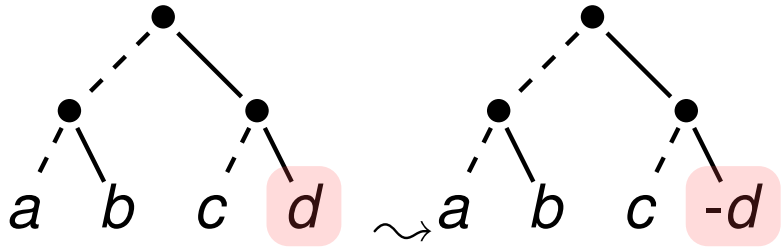

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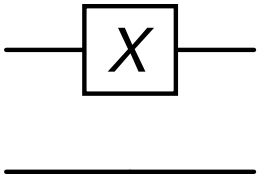

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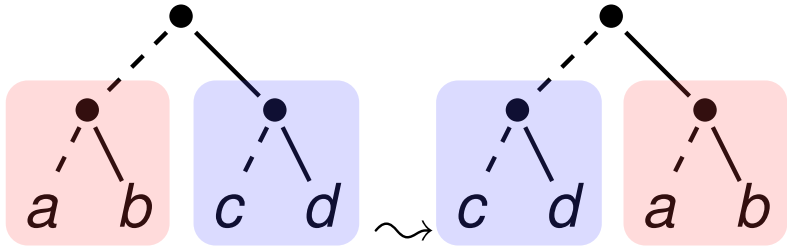

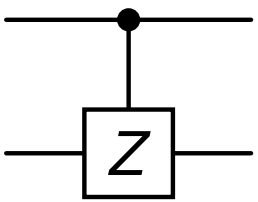

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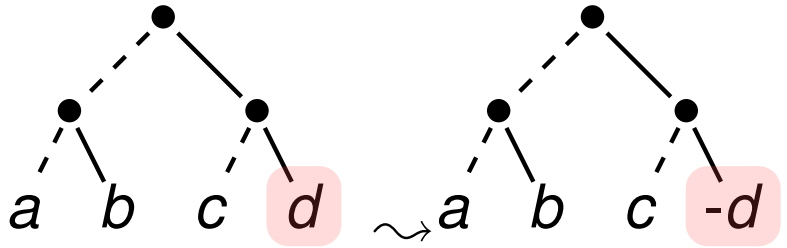

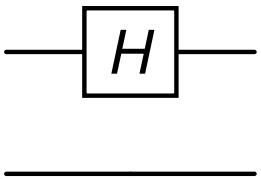

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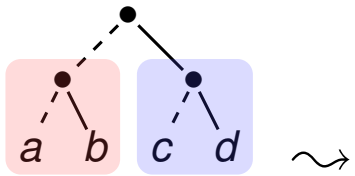

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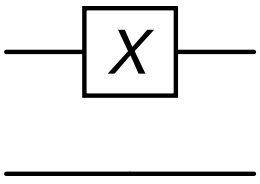

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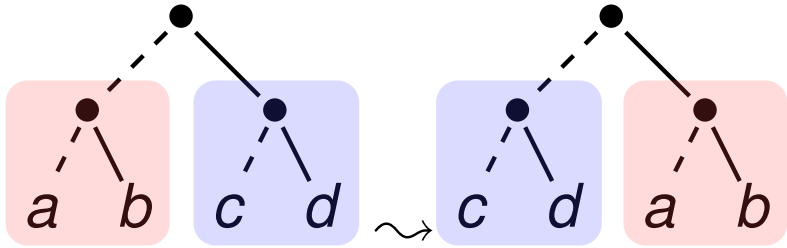

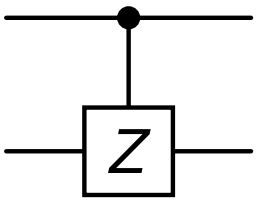

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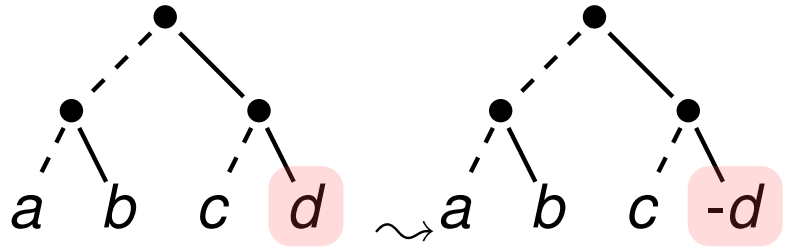

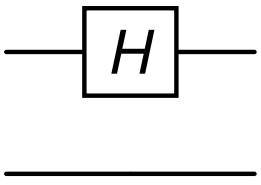

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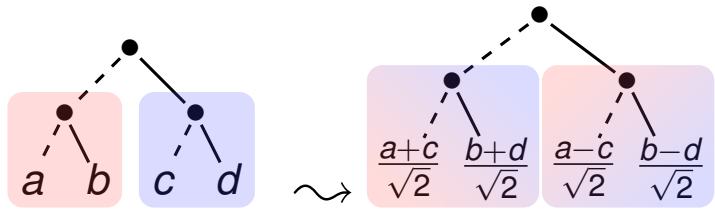

Hadamard gate

Sets of Quantum States are Sets of Trees

- How to efficiently represent sets of trees?

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Tree automata!

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  - finite-state automata representing sets of finite trees
  - extension of standard finite automata for regular languages

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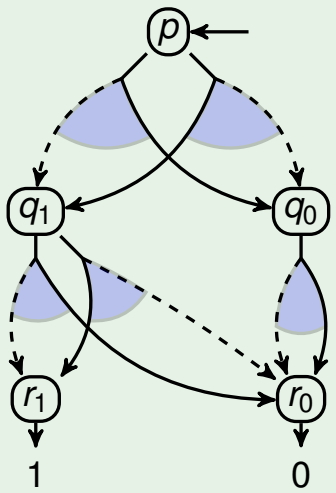
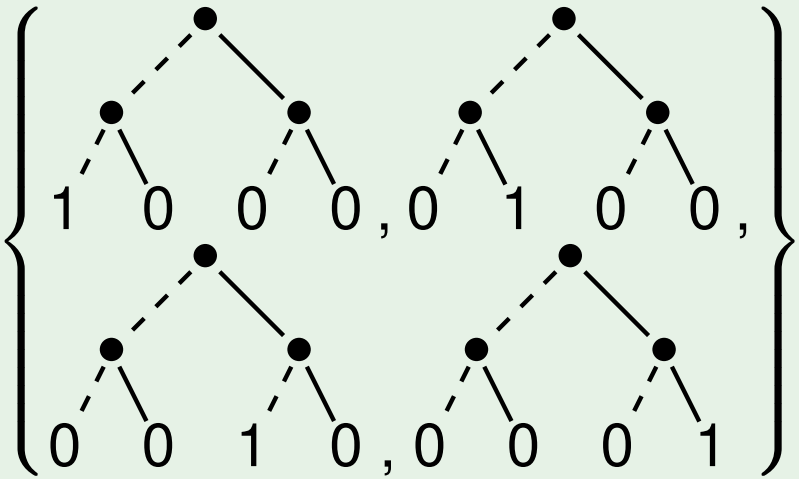
Example



represents the set



Representing *Pre* and *Post* with Tree Automata

$$\underset{\text{precondition}}{\{\mathcal{A}_{Pre}\}} \quad \underset{\text{circuit}}{C} \quad \underset{\text{postcondition}}{\{\mathcal{A}_{Post}\}}$$

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precondition — *postcondition*

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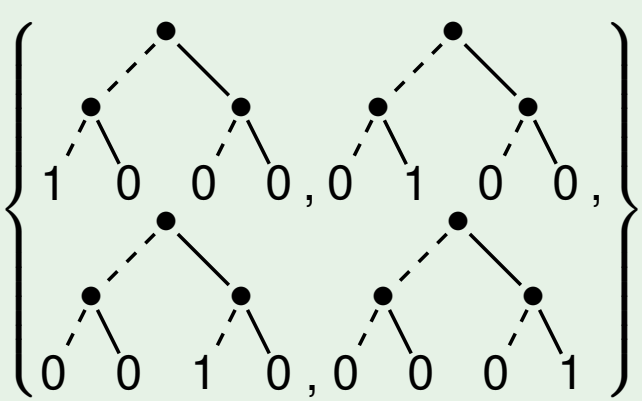
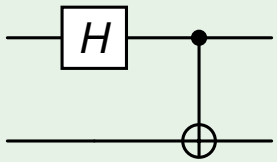
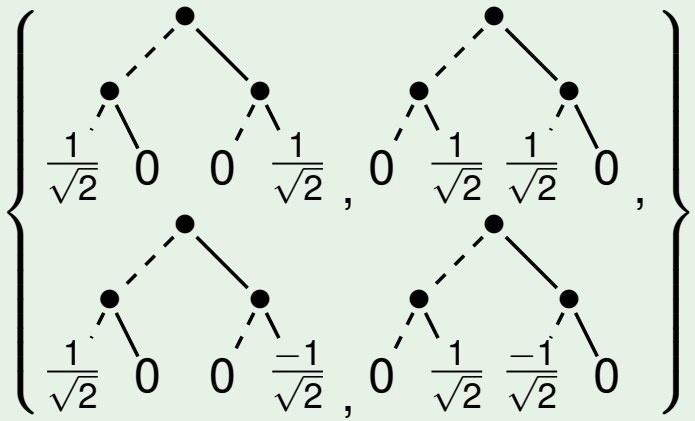


$\mathcal{L}(\mathcal{A}_{Pre})$





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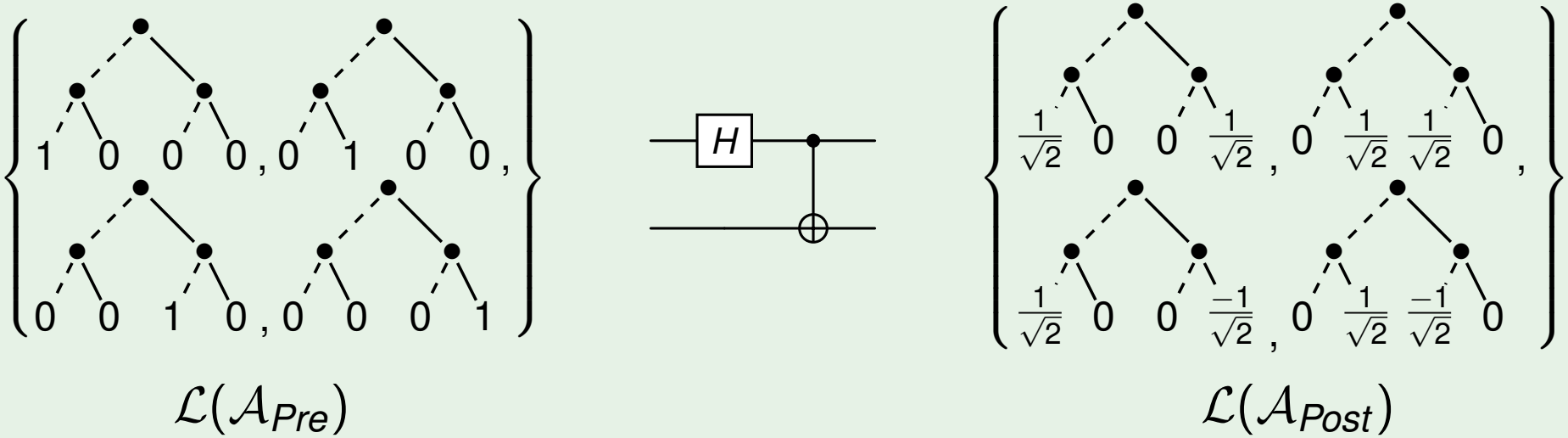


$\mathcal{L}(\mathcal{A}_{Pre})$ $\mathcal{L}(\mathcal{A}_{Post})$

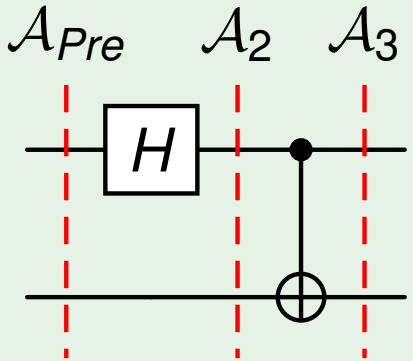
- $\mathcal{A}$'s size can be small
  - e.g., $\mathcal{A}$ for $\{|w\rangle : w \in \{0,1\}^n\}$ needs $\mathcal{O}(n)$ states/transitions

Verification with Tree Automata

$$\underset{\text{precondition}}{\{\mathcal{A}_{Pre}\}} \quad \underset{\text{circuit}}{C} \quad \underset{\text{postcondition}}{\{\mathcal{A}_{Post}\}}$$

- Run C with $\mathcal{A}_{Pre}$:

Example



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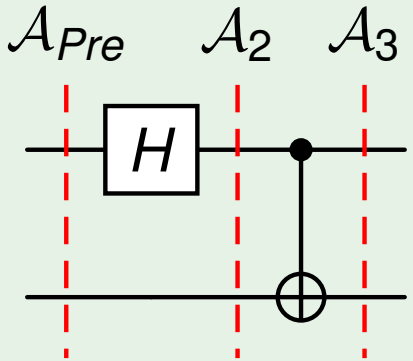
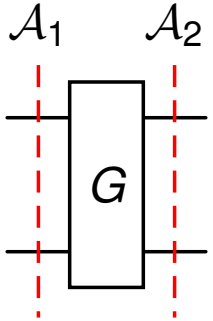


- . . . and test $\mathcal{L}(\mathcal{A}_3) \subseteq \mathcal{L}(\mathcal{A}_{Post})$
  - tree automata inclusion is decidable (**EXPTIME**-complete)

Abstract Transformers for Quantum Gates



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  - naively (i.e., one tree by one) — doesn't scale

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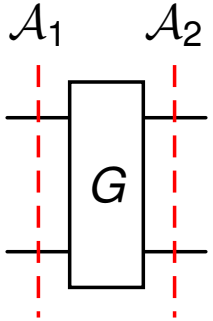

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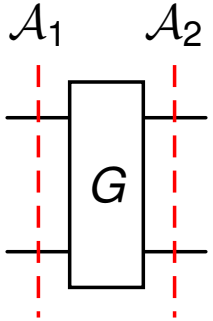

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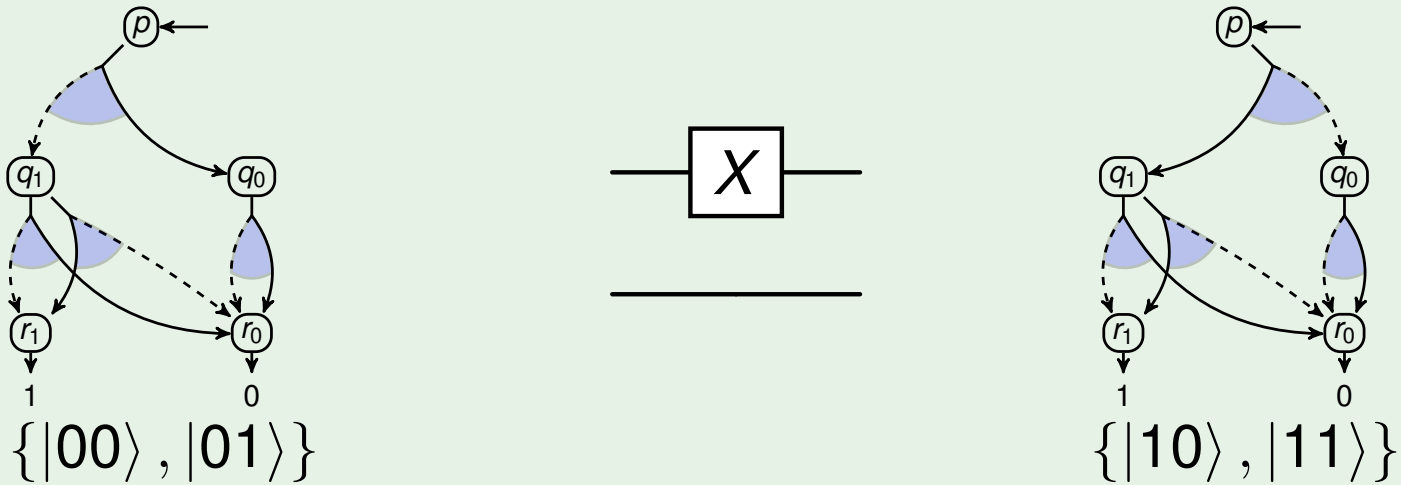

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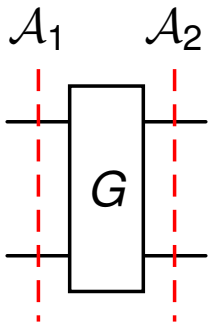

- Supported gate types:
  - (anti-)diagonal: X, Y, Z, S, T, R_z, controls (*CNOT*, *CZ*, *Toffoli*, ...)
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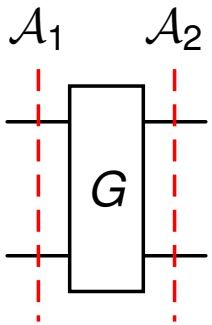

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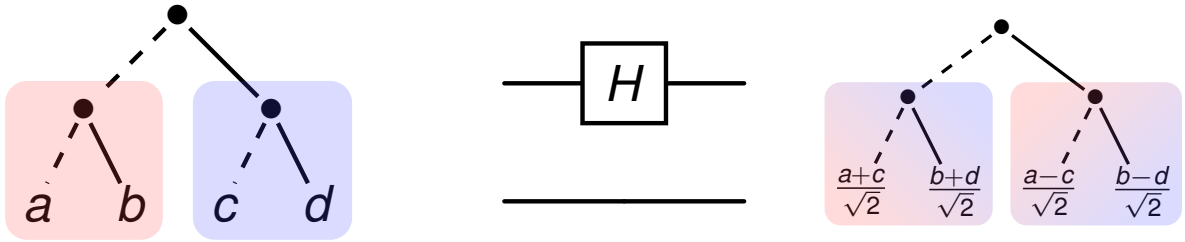
    - simple manipulation with automaton: $\mathcal{O}(|\mathcal{A}_1|)$
  - general: $H, R_x, R_y, \ldots$
    - need to synchronize subtrees of the same tree



    - qubit reorder $\rightarrow$ leaf operation $\rightarrow$ qubit reorder: $\mathcal{O}(2^{|\mathcal{A}_1|})$

Quantum Circuit Verification Algorithm

$$\{\mathcal{A}_{Pre}\}\ \underset{circuit}{C}\ \{\mathcal{A}_{Post}\}$$

precondition *postcondition*

- Algorithm:
  1. Start with $\mathcal{A}_{Pre}$.

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- Scales to up to 40 qubits / 140k gates.
- Found a confirmed bug in QCEC (SOTA equivalence checker).

[Chen, Chung, Lengál, Lin, Tsai, Yen. An Automata-Based Framework for Verification and Bug Hunting in Quantum Circuits. PLDI'23/CACM RH]

New Automata Models

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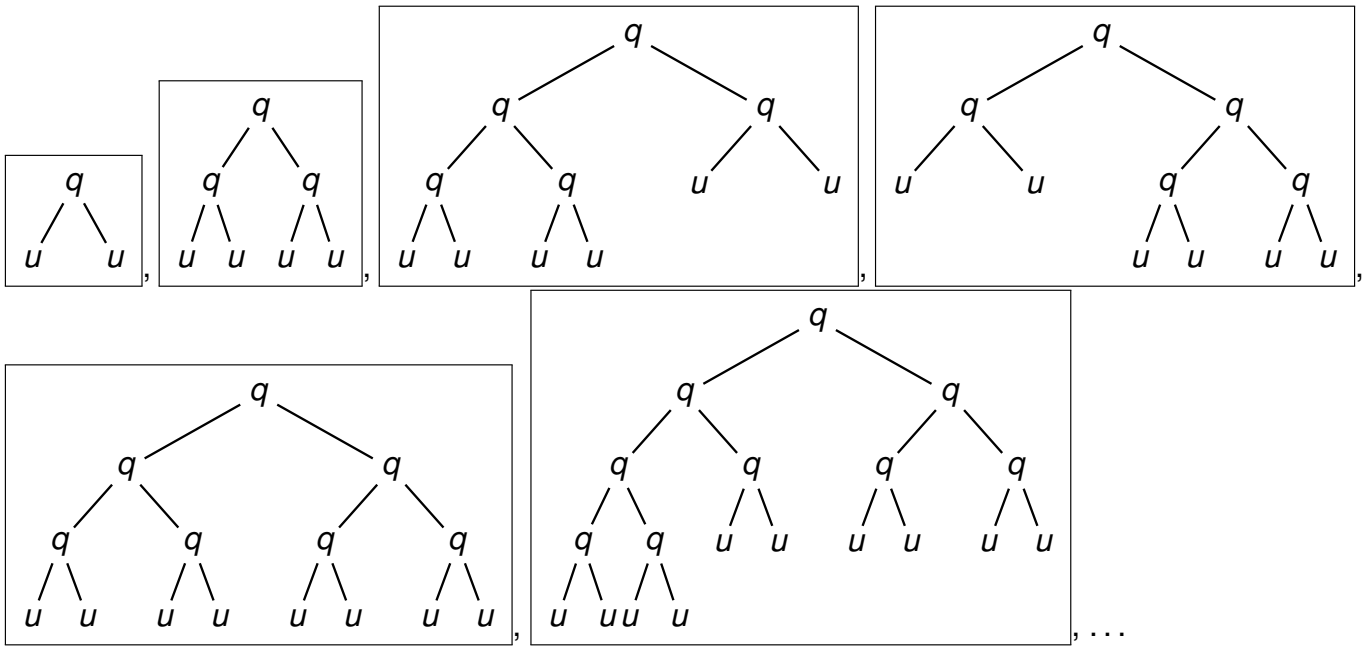
q is root, u is leaf



Level-Synchronized Tree Automata (LSTAs)

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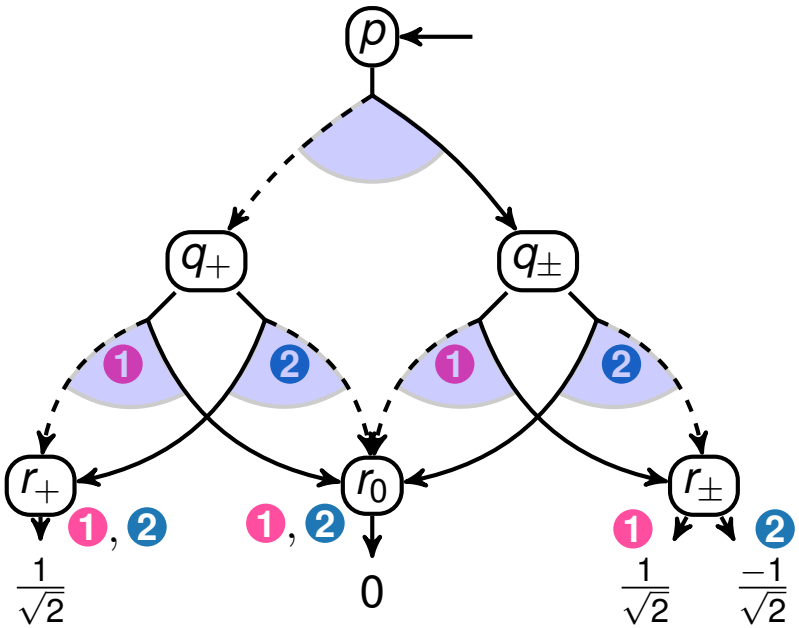
- allow synchronization across subtrees rooted at the same level

$q_+ \xrightarrow{1} (r_+, r_0)$ $\quad$ $q_+ \xrightarrow{2} (r_0, r_+)$

$q_\pm \xrightarrow{1} (r_0, r_\pm)$ $\quad$ $q_\pm \xrightarrow{2} (r_\pm, r_0)$



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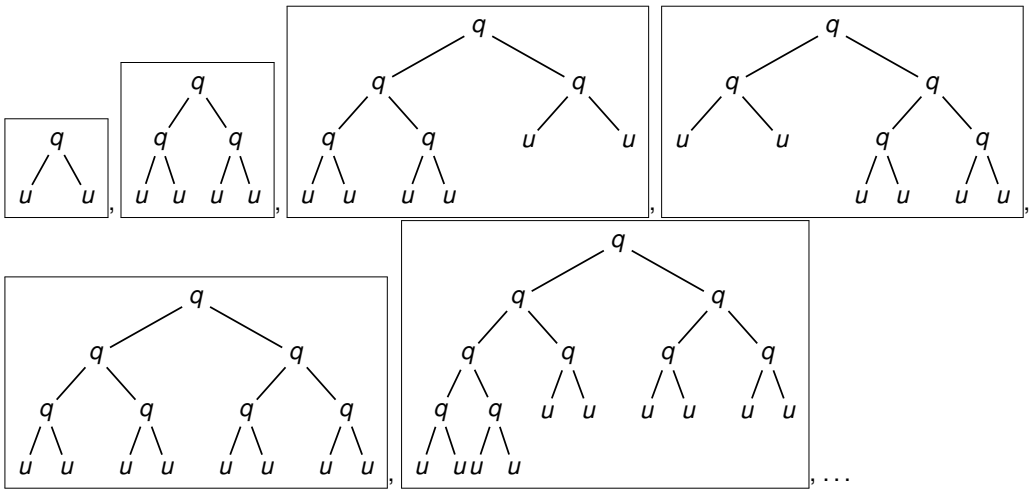

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With synchronization

$$q \xrightarrow{1} (q, q)$$

$$q \xrightarrow{2} (u, u)$$

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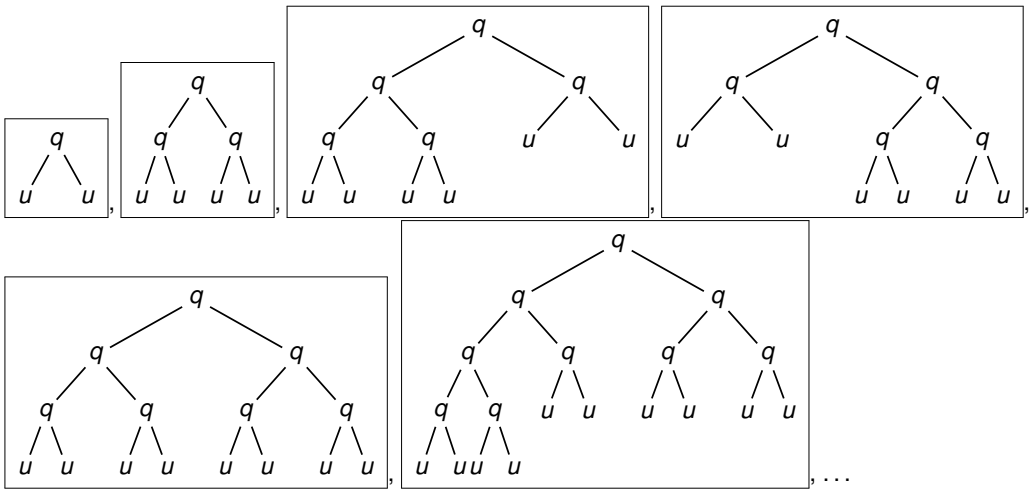
On every level, transitions with the same colour need to be taken!



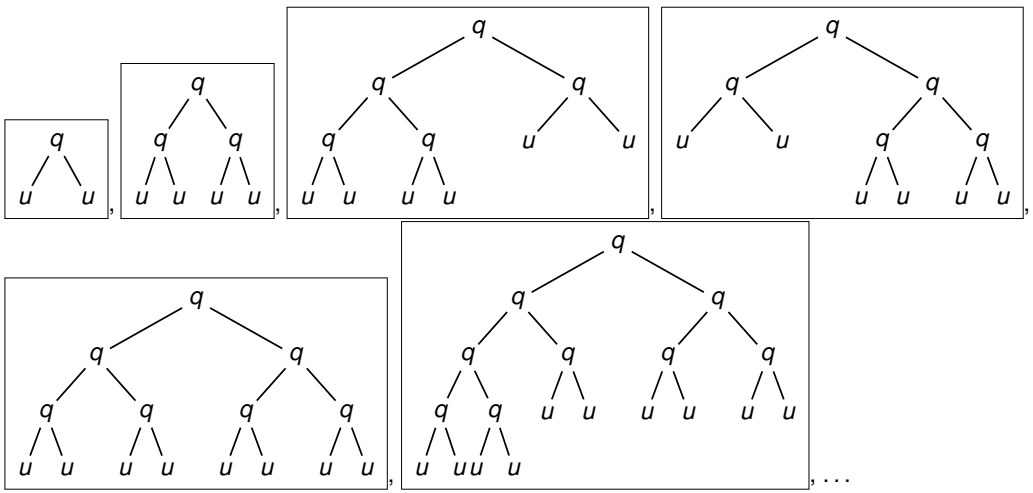
Level-Synchronized Tree Automata (LSTAs)

No synchronization (standard TAs)

$q \to (q, q)$

$q \to (u, u)$

q is root, u is leaf



With synchronization

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$q \xrightarrow{2} (u, u)$

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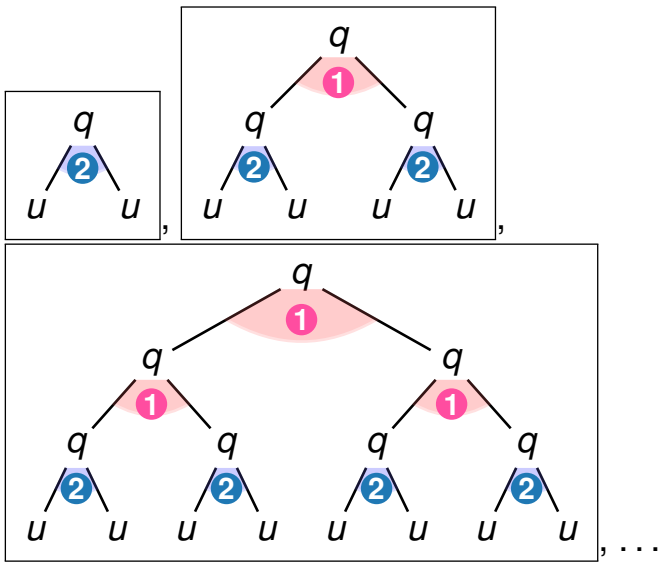
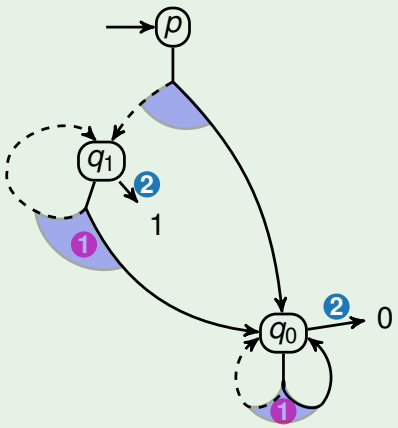
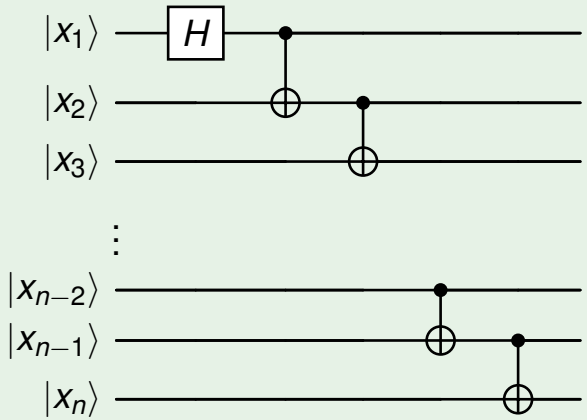


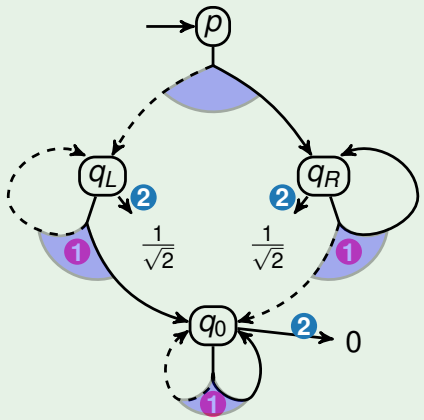
Level-Synchronized Tree Automata (LSTAs)

Enable basic parameterized verification

Example (GHZ)







$\{|0^n\rangle : n \geq 1\}$

$\{\frac{1}{\sqrt{2}}|0^n\rangle + \frac{1}{\sqrt{2}}|1^n\rangle : n \geq 1\}$

Level-Synchronized Tree Automata (LSTAs)

Provide better scalability

- cost of operations
  - (anti-)diagonal gates: still $\mathcal{O}(|\mathcal{A}|)$
  - general gates: $\mathcal{O}(|\mathcal{A}|^2)$ (improved from $\mathcal{O}(2^{|\mathcal{A}|})$)

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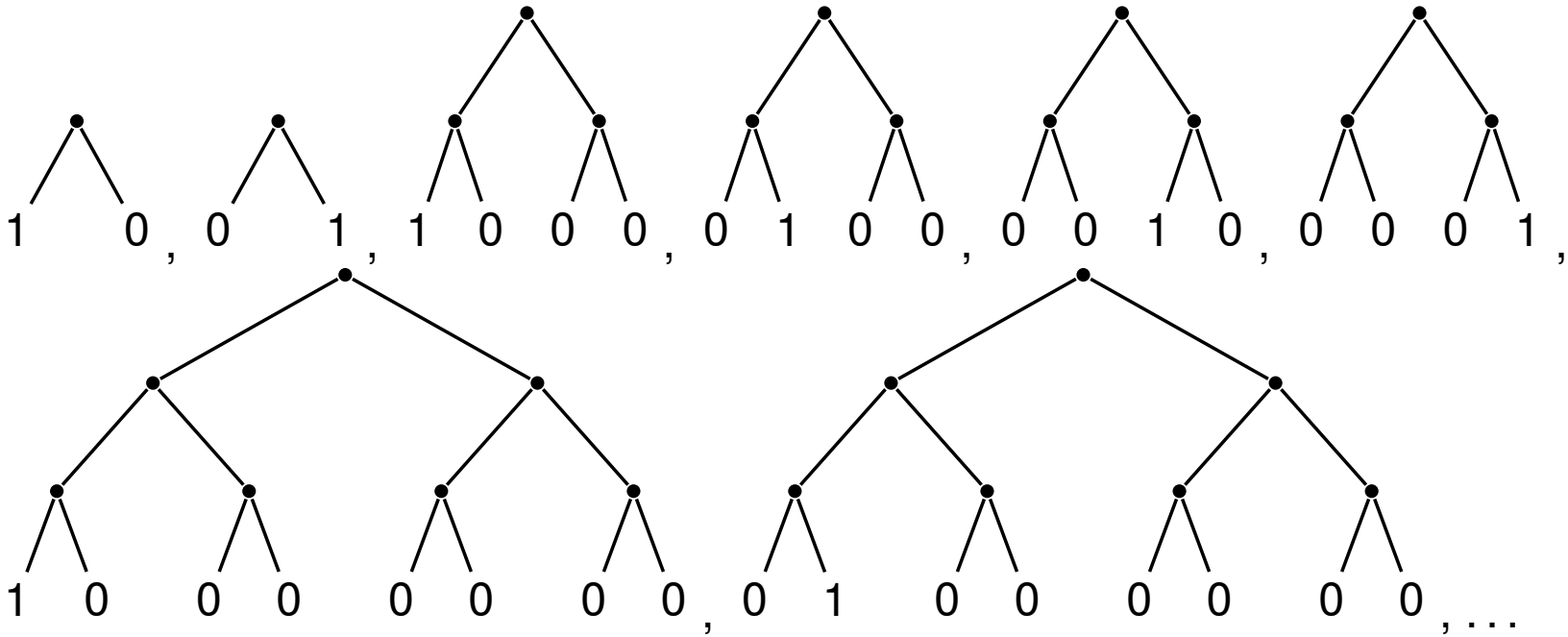
- real-world speed-ups:
  - 4 min 45 s $\rightsquigarrow$ 0.0 s (BV-All, 27 qbits)
  - $>$5 min $\rightsquigarrow$ 0.6 s (GHZ-All, 128 qbits)
  - $>$5 min $\rightsquigarrow$ 39.3 s (Grover-All, 30 qbits)
  - $>$5 min $\rightsquigarrow$ 17.1 s (Grover-Iter, 200 qbits)

[Abdulla, Chen, Chen, Holík, Lengál, Lin, Lo, Tsai. Verifying Quantum Circuits with Level-Synchronized Tree Automata. POPL'25.]

Level-Synchronized Tree Automata (LSTAs)

- LSTAs can express all computational basis states



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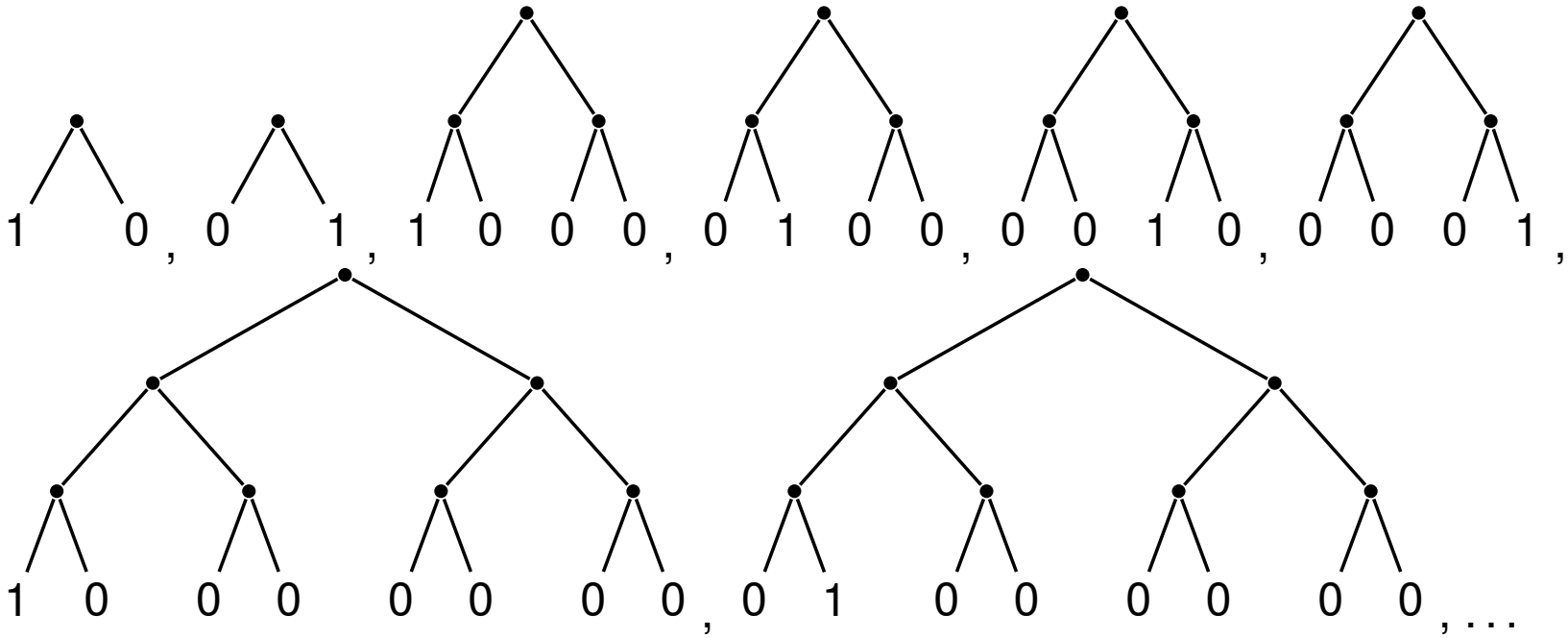
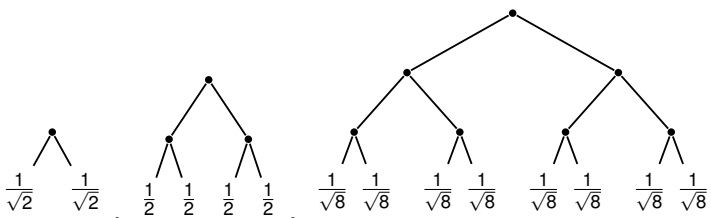


- but still cannot express all uniform superposition states
  - ∞-many amplitudes
  - important for parameterized verification!



Parameterized Quantum Circuits

- Circuits are parameterized by their size

Parameterized Quantum Circuits

- Circuits are parameterized by their size
- Many standard quantum circuits are parameterized
  - generalized GHZ
  - Bernstein-Vazirani
  - Grover's search
  - error-correction
  - arithmetic circuits
  - quantum counting
  - quantum phase estimation
  - quantum Fourier transform,
  - Shor's algorithm, ...

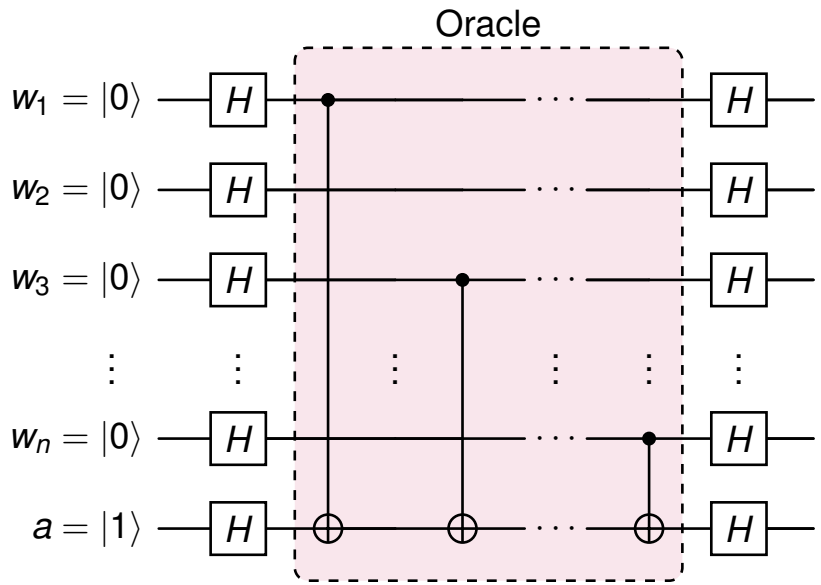


Figure: Bernstein-Vazirani for the secret $(10)^*(1+\varepsilon)$

Synchronized Weighted Tree Automata (SWTAs)

Synchronized Weighted Tree Automata
LSTAs + weights + alternation

[1] we ignore internal labels and only consider the structure of trees and leaf values

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LSTAs + weights + alternation

$\mathcal{A} = \langle Q, \Omega, \delta, \mathit{root}, E \rangle$[1]

- $Q = \{q_1, \ldots, q_n\}$ — a finite set of states,
- $\Omega = \{1, \ldots, k\}$ — a finite set of colours (for synchronization),
- δ — a transition function (details later),
- $\mathit{root} \in Q$ — the root state, and
- $E \subseteq Q$ — set of leaf states.

[1] we ignore internal labels and only consider the structure of trees and leaf values

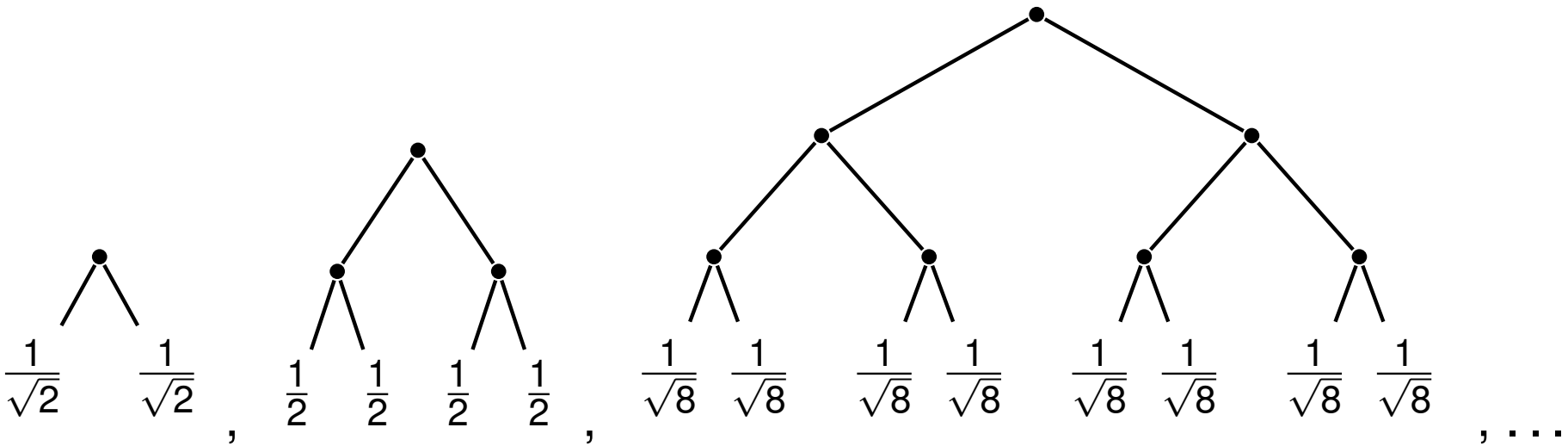
Synchronized Weighted Tree Automata

Example:

$$root \xrightarrow{1} \left(\frac{1}{\sqrt{2}}p, \quad \frac{1}{\sqrt{2}}p\right)$$
$$p \xrightarrow{1} \left(\frac{1}{\sqrt{2}}p, \quad \frac{1}{\sqrt{2}}p\right)$$

- p is a leaf state

- expresses a set of uniform superposition states:



Synchronized Weighted Tree Automata

General form of transitions:

$$u \xrightarrow{1} \left(\tfrac{1}{2}p + \tfrac{1}{\sqrt{8}}q, \quad \tfrac{1}{\sqrt{2}}t - \tfrac{1}{4}z\right)$$

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1. generate trees t_p and t_q from p and q (can be the leaf $\boxed{1}$ if a state is leaf)
2. multiply leaves of t_p by $\frac{1}{2}$ and leaves of t_q by $\frac{1}{\sqrt{8}}$
3. $t_{left} \leftarrow \frac{1}{2}t_p + \frac{1}{\sqrt{8}}t_q$ (structures of t_p and t_q need to match)
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language $\mathcal{L}(\mathcal{A})$: the set of trees generated by $\mathcal{A}$

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- intersection non-emptiness: undecidable (reduction from PCP)
  - language intersection: not-closed (from the above)
- language inclusion/equivalence: undecidable (reduction from emptiness of TM)
  - $\rightsquigarrow$ bad (?) $\qquad C(\mathit{Pre}) \stackrel{?}{\subseteq} \mathit{Post}$

Synchronized Weighted Tree Automata — Properties

alternative:

- don't talk about the standard language, but
- $\mathcal{A}$'s tree function $f_{\mathcal{A}} : \Omega^* \to \mathbb{T}_{\mathbb{C}}$ ($\Omega = \{1, \ldots, k\}$, $\mathbb{T}_{\mathbb{C}} =$ perfect trees with $\mathbb{C}$ leaves)

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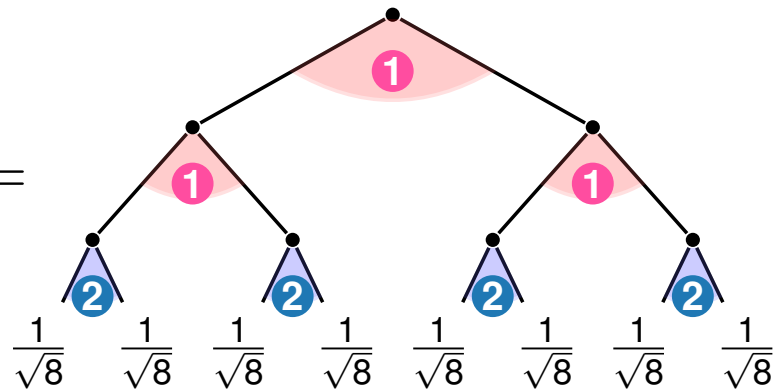

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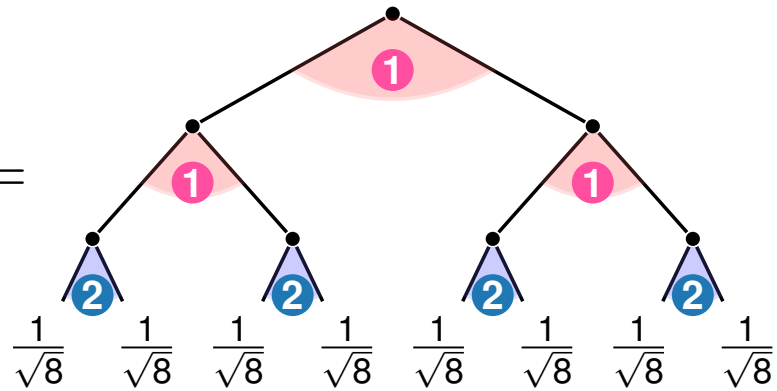

- very useful!
  - $\rightsquigarrow$ each computational basis state tree can map to one string in Ω^*: $Pre \leftrightarrow \Omega^*$
  - $C(Pre)$ preserves tree function: e.g., $f_{Pre}(112) \xrightarrow{C} f_{C(Pre)}(112)$
  - allows relational specification, circuit equivalence checking

Synchronized Weighted Tree Automata — Properties

- tree function equivalence/inclusion of $\mathcal{A}$ and $\mathcal{B}$

$$f_{\mathcal{A}} \stackrel{?}{=} f_{\mathcal{B}} \qquad\qquad f_{\mathcal{A}} \stackrel{?}{\subseteq} f_{\mathcal{B}}$$

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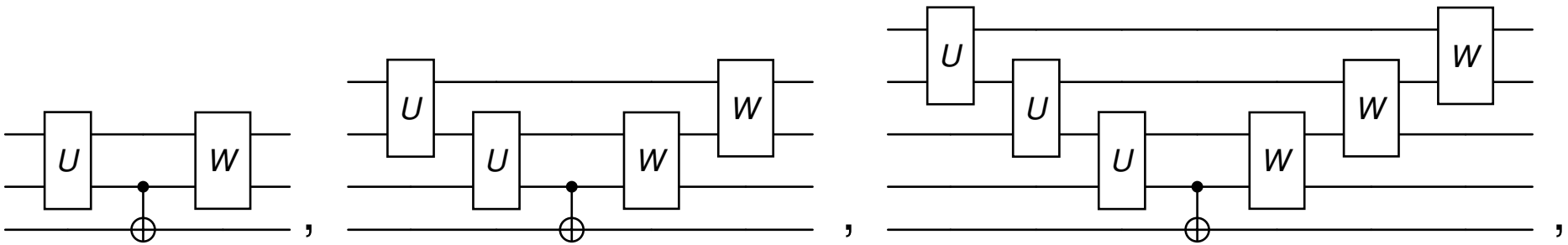
  5. run Karr's algorithm to compute for every state in $\mathcal{S}$ the vector space of all linear relations, check 0-dim subspace for interesting states
     - M. Karr. Affine Relationships Among Variables of a Program. Acta Inf., 6:133–151, 1976.

Weighted Tree Transducers (WTTs)

Issue: How to (efficiently) represent size-parameterized families of circuits, e.g.,



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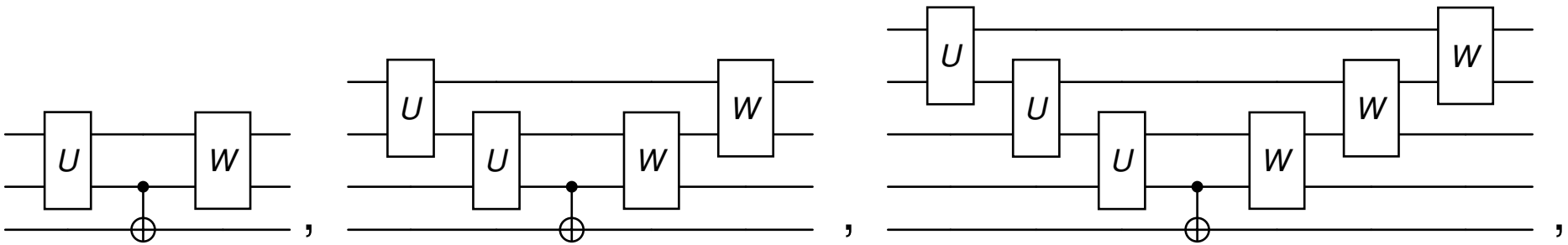

...

- previous approach (specialized automata operations) doesn't work any more

Regular Model Checking

- popular approach to parameterized verification of classical systems

$$C(Pre) \stackrel{?}{\subseteq} Post$$

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Goal

We need a transducer model for SWTAs!

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- **L** and **R** denote the input **L**eft and **R**ight sub-trees

Weighted Tree Transducers (WTTs)

- capture compactly all fixed-sized gates
  - previous work: specialized procedures
- composition: quadratic
- image computation: quadratic
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- WTT for fixed-sized QFT: quadratic to #qubits
  - QFT: Quantum Fourier Transform

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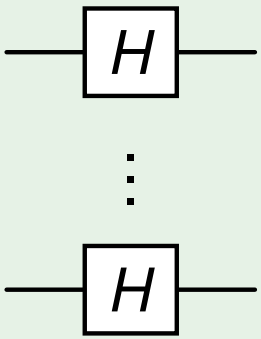
Example (Hadamard Transform)



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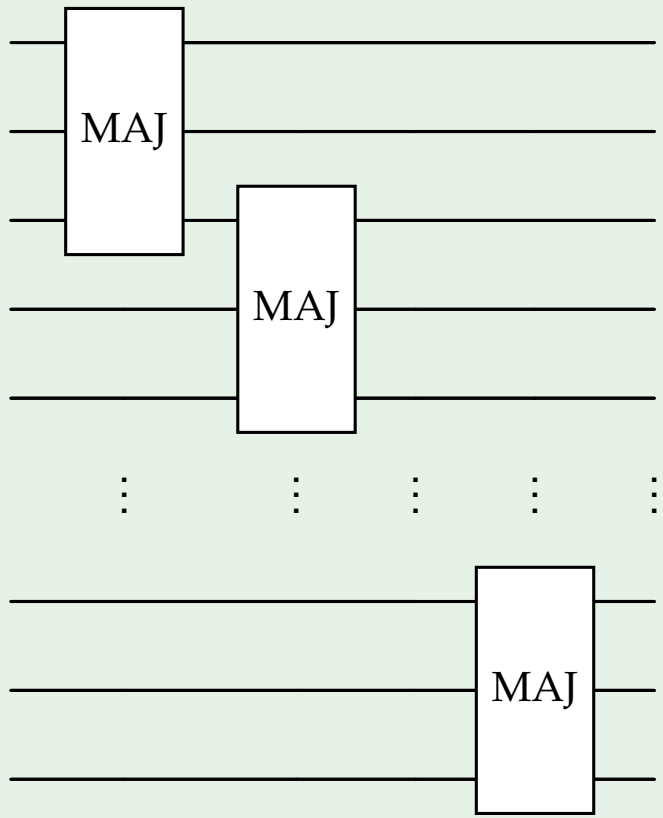
Example (Part of Ripple-Carry Adder)



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Implementation: prototype AUTOQ-PARA

Benchmarks:

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Ham. sim.	0.663 s

Equivalence testing

- Grover — one Grover iter., w/ vs. w/o multi-control gates
- Ham. sim. — Hamiltonian simulation of a *1D Heisenberg spin chain*, basic vs. optimized version

[Abdulla, Chen, Hečko, Holík, Lengál, Lin, Thinniyam. Parameterized Verification of Quantum Circuits. POPL'26.]

Takeaways and Future Directions

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Quantum ♡ Automata

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Thank you!